PARTIAL DIFFERENTIAL EQUATIONS
AN INTRODUCTION

A.D.R. Choudary, Saima Parveen
Constantin Varsan

First Edition

Abdus Salam School of Mathematical Sciences, Lahore, Pakistan.

A.D.Raza Choudary
Abdus Salam School of Mathematical Sciences,
GC University, Lahore, Pakistan.
choudary@cwu.edu

Saima Parveen
Abdus Salam School of Mathematical Sciences,
GC University, Lahore, Pakistan.
saimashaa@gmail.com

Constatin Varsan
Mathematical Institute of Romanian Academy
Bucharest, Romania.
constantin.varsan@imar.ro

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To our families

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Introduction

This book is encompassing those mathematical methods used in describing and solving second order partial differential equation (PDE) of elliptic, hyperbolic and parabolic type.

Our priority is to make this difficult subject accessible to those interested in applying mathematics using differential equations as models. It is accomplished by adding some fundamental results from ordinary differential equations (ODE) regarding flows and their differentiability properties which are useful in constructing solution of Hamilton Jacobi equations.
The analysis of first order Cauchy-Kowalevska system is limited to their application in constructing analytical solution for hyperbolic and elliptic equations which are frequently used in Physics and Mechanics.
The exposition is subjected to a gradually presentations and the classical methods called as Fourier, Riemann, Fredholm integral equations and the corresponding Green functions are analyzed by solving significant examples.
The analysis is not restricted to the linear equations. Non linear parabolic equation or elliptic equations are included enlarging the meaning of the weak solution.
This university text includes a Lie-geometric analysis of gradient systems of vector fields and their algebraic representation with a direct implication in treating both first order overdetermined systems with singularities and solutions for first order PDE.
We are aware that a scientific presentation of PDE must contain an additional text introducing several results from multidimensional analysis and it is accomplished including Gauss-Ostrogadsky formula, variational methods of deriving PDE and recovering harmonic functions from its boundary values (see appendices I,II,III of chapter III).
Each chapter of this book includes exercises and problems which can be solved provided the given hints are used.
Partially, some subjects of this university-text have been lectured at Abdus Salam School of Mathematical Sciences (ASSMS), Lahore and it is our belief that this presentation deserve a wider distribution among universities having graduate program in mathematics.
The authors express their gratitude to Abdus Salam School of Mathematical Sciences (ASSMS) of GC University Lahore for their encouragements and assistance in writing

this book.
This book also includes two subjects that encompass good connection of PDE with differential geometry and stochastic analysis. The first subject is represented by gradient systems of vector fields and their influence in solving first order overdetermined systems. The second subject introduces approximations of SDE by ODE which can be meaningful in deriving significant results of stochastic partial differential equations($SPDE$) represented here by stochastic rule of derivation.Our belief is that such subjects must be presented in any introductory monography treating PDE.

Chapter 1

Ordinary Differential Equations(ODE)

1.1 Linear System of Differential Equations

A linear system of differential equations is described by the following scalar differential equations

$$\frac{dy^i}{dx} = \sum_{j=1}^{n} a_{ij}(x)y^j + b_i(x), x \in I \subseteq \mathbb{R}, i = 1, 2, ..., n \tag{1.1}$$

where the scalar functions a_{ij} and b_i are continuous on the interval $I \subseteq \mathbb{R}\,(a_{ij}, b_i \in \mathbb{C}(I;\mathbb{R}))$. Denote $z = col(y^1, \ldots, y^n) \in \mathbb{R}^n$ as the column vector of the unknown functions and rewrite (1.1) as a linear vector system

$$\frac{dz}{dx} = A(x)z + b(x), x \in I \subseteq \mathbb{R}, z \in \mathbb{R}^n \tag{1.2}$$

where $A = (a_{ij})$, $(i, j) \in \{1, \ldots, n\}$ stands for the corresponding $n \times n$ continuous matrix valued function and $b = \mathrm{col}(b_1, \ldots, b_n) \in \mathbb{R}^n$ is a continuous function on I valued in $\mathbb{R}^n$. A solution of the system (1.2) means a continuous function $z(x)$: $I \to \mathbb{R}^n$ which is continuously differentiable ($z \in \mathcal{C}^1(I;\mathbb{R}^n)$) and satisfies (1.2) for any $x \in I$, i.e $\frac{dz}{dx}(x) = A(x)z(x) + b(x)$, for all $x \in I$. The computation of a solution as a combination of elementary functions and integration is not possible without assuming some particular structures regarding the involved matrix A. As far as some qualitative results are concerned (existence and uniqueness of a solution) we do not need to add new assumptions on the A and a Cauchy problem solution receives a positive answer.

For a fixed pair $(x_0, z_0) \in I \times \mathbb{R}^n$ we define $\{z(x) : x \in I\}$ as the solution of the system (1.2) satisfying $z(x_0) = z_0$ (Cauchy problem solution). To prove uniqueness of a Cauchy problem solution we recall a standard lemma.

Lemma 1.1.1. *(Gronwall) Let $\varphi(x), \alpha(x) : [a, b] \to [0, \infty)$ and a constant $M \geqslant 0$ be given such that the following integral inequality is valid*

$$\varphi(x) \leqslant M + \int_a^x \alpha(t)\varphi(t)dt, x \in [a, b]$$

where α and φ are continuous scalar functions. Then

$$\varphi(x) \leqslant M\,exp \int_a^b \alpha(t)dt, \forall\, x \in [a, b]$$

Proof. Denote $\psi(x) = M + \int_a^b \alpha(x)\varphi(t)dt$ and a straight computation lead us to $\varphi(x) \leqslant \psi(x), \alpha \in [a, b]$ and

$$\begin{cases} \frac{d\psi(x)}{dx} = & \alpha(x)\varphi(x) \leqslant \alpha(x)\psi(x),\ x \in [a, b] \\ \psi(a) = & M \end{cases} \tag{1.3}$$

The differential inequality in (1.3) can be written as a scalar equation

$$\begin{cases} \frac{d\psi(x)}{dx} = & \alpha(x)\psi(x) + \delta(x),\ x \in [a, b] \\ \psi(a) = & M \end{cases} \tag{1.4}$$

where $\delta(x) \leqslant 0, \forall\, x \in\ [a, b]$. Using the integral representation of of the scalar equation solution we get

$$\psi(x) = [\exp \int_a^x \alpha(t)dt]\{M + \int_a^x (\exp - \int_a^t \alpha(s)ds)\delta(t)dt\}$$

and as consequence(see $\delta(x) \leqslant 0, x \in [a, b]$)we obtain

$$\varphi(x) \leqslant \psi(x) \leqslant M[\exp \int_a^b \alpha(t)dt], \forall\, x \in [a, b]$$

□

Theorem 1.1.2. *(Existence and uniqueness of Cauchy problem solution)*
Let $A(x) : I \to M_{n\times n}(n \times n$ matrices) and $b(x) : I \to \mathbb{R}^n$ be continuous functions and $x_0 \in I \subseteq \mathbb{R}^n$ be fixed. Then there exists a unique solution $z(x) : I \to \mathbb{R}^n$ of the system satisfying the initial condition $z(x_0) = z_0$ (Cauchy condition).

Proof. We notice that a solution $\{z(x) : x \in I\}$ of (1.2) satisfying $z(x_0) = z_0$ fulfils

the following integral equation

$$z(x) = z_0 + \int_{x_0}^{x} [A(t)z(t) + b(t)]dt, \forall x \in I \tag{1.5}$$

and conversely any continuous function $z(x) : I \to \mathbb{R}^n$ which satisfies (1.5) is continuously differentiable
and

$$\frac{dz}{dx} = A(x)z + b(x), \forall\, x \in I,\ z(x_0) = z_0$$

We suggest to look for a continuous function satisfying integral equation (1.5) and it is accomplished using Picard's iteration method which involves the following sequence of continuous functions $\{z_k(x) : x \in I\}_{k \geqslant o}$

$$z_0(x) = z_0, z_{k+1}(x) = z_0 + \int_{x_0}^{x} [A(t)z_k(t) + b(t)]dt \tag{1.6}$$

For each compact interval $J \subseteq I, x_0 \in J$, the sequence $\{z_k(x) : x \in J\}_{k \geqslant o} \subseteq \mathcal{C}(J; \mathbb{R}^n)$ (Banach space) is a Cauchy sequence in a Banach space, where

$$z_{k+1}(x) = z_0 + u_1 + u_2 ... + u_{k+1}, u_{j+1}(x) = z_{j+1}(x) - z_j(x), j \geqslant 0 \tag{1.7}$$

In this respect, $\{z_k(x) : x \in J\}$ rewritten as in (1.7) coincides with a partial sum of the following series

$$\sum(x) = z_0 + u_1 + u_2 ... + u_k + ..., x \in J \tag{1.8}$$

and the convergence of $\{\sum(x) : x \in J\} \subseteq \mathcal{C}(J; \mathbb{R}^n)$ can be obtained using a convergent numerical series as an upper bound for it. The corresponding convergent numerical series is of exponential type and it is contained in the following estimate. Using standard induction argument we prove

$$\mid u_{j+1}(x) \mid \leqslant (1+ \mid z_0 \mid) \frac{k^{j+1} |x - x_0|^{j+1}}{(j+1)!}, \forall j \geqslant 0, x \in J \tag{1.9}$$

where

$$k = \max(\max_{t \in J} \mid A(t) \mid, \max_{t \in J} \mid b(t) \mid)$$

For $j = 0$ we see easily that $\mid u_1 \mid \leqslant (1+ \mid z_0 \mid)k \mid (x - x_0) \mid, x \in J$, and using

$$u_{j+2}(x) = z_{j+2}(x) - z_{j+1}(x) = \int_{x_0}^{x} A(t)u_{j+1}(t)dt$$

we get

$$\mid u_{j+2}(x) \mid \leqslant k \int_{x_0}^{|(x-x_0)|} \mid u_{j+1}(t) \mid dt \leqslant (1+ \mid z_0 \mid) \frac{k^{j+2} |(x - x_0)|^{j+2}}{(j+2)!}, x \in J \tag{1.10}$$

provided $\{u_{j+1}(x) : x \in J\}$ satisfies (1.9). As a consequence, the inequality (1.9)

show us that

$$| u_{j+1}(x) | \leqslant (1+ | z_0 |)\frac{k^{j+1} | J |^{j+1}}{(j+1)}, \forall j \geqslant 0, x \in J \tag{1.11}$$

where $| J |= \max_{x\in J} | x - x_0 |$and the corresponding convergent numerical series is given by $(1+ | z_0 |)\exp k | J |$, where the constant $k \geqslant 0$ is defined in (1.9). Using(1.11) into (1.8) we obtain that sequence$\{z_k(x) : x \in I\}_{k\geqslant o}$ defined in (1.6) is uniformly convergent to a continuous function $\{z_j(x) : x \in J\}$ and by passing $k \to \infty$ in (1.6) we get

$$z_j(x) = z_0 + \int_{x_0}^{x}[A(t)z_j(t) + b(t)]dt, \forall x \in J \tag{1.12}$$

It shows that $\{z_j(x) : x \in J\}$ is continuously differentiable, $z_j(x_0) = z_0$ and satisfies (1.2) for any $x \in J \subseteq I$. Define $\{\widehat{z}(x) : x \in I\}$ as a "inductive limit" of the sequence $\{z_m(x) = z_{j_m}(x) : x \in J_m\}_{m\geqslant 1}$
where

$$I = \bigcup_{m=1}^{\infty} J_m, J_{m+1} \supseteq J_m, x_0 \in J_m$$

Then $\widehat{z}(x) = z_m(x), x \in J_m, m \geqslant 1$ is continuously differentiable function satisfying integral equation (1.5).

Uniqueness

It will be proved by contradiction and assuming another solution $\{z_1(x) : x \in I\}$ of (2)exits such that $z_1(x_0) = z_0$ and $z_1(x^*) \neq \tilde{z}(x^*)$ for some $x^* \in I$ then $u(x) = \widehat{z}(x) - z_1(x), x \in I$, verifies the following linear system

$$u(x) = \int_{x_0}^{x} A(t)u(t)dt, x \in I, \text{where}\{\widehat{z}(x), x \in I\} \text{ is defined above}. \tag{1.13}$$

Let $J \subseteq I$ be a compact interval such that $x_0, x^* \in I$ and assuming that $x_0 \leqslant x^*$ we get

$$\varphi(x) =| u(x) |\leqslant \int_{x_0}^{x_0+|x-x_0|} \alpha(t)\varphi(t)dt, \ \forall x \in [x_0, x^*] \tag{1.14}$$

where$\alpha(t) =| A(t) |\geqslant 0$ and $\varphi(t) =| u(t) |$ are continuous scalar functions. Using Lemma 1.1.1 with$M = 0$ we obtain $\varphi(x) = 0$ for any $x \in [x_0, x^*]$ contradicting $\varphi(x^*) > 0$. The proof for $x_0 > x^*$ is similar. The proof is complete. □

1.2 Fundamental Matrix of Solution

A linear homogenous system is described by

$$\frac{dz}{dx} = A(x)z, z \in \mathbb{R}^n \tag{1.15}$$

where the $(n \times n)$ matrix $A(x), x \in I$ is a continuous mapping valued in $M_{n\times n}$. Let $x_0 \in I$ be fixed and denote $\{z_i(x) : x \in I\}$ the unique solution of (1.15) satisfying the Cauchy condition $z_i(x_0) = e_i \in \mathbb{R}^n, i \in \{1, 2, ..., n\}$
where $\{e_1, ..., e_n\} \subseteq \mathbb{R}^n$ is the canonical basis. Denote

$$C(x; x_0) = \parallel z_1(x)....z_n(x) \parallel, x \in I \tag{1.16}$$

where $z_i(x) \in \mathbb{R}^n$ is a column vector. The matrix $C(x; x_0), x \in I$ defined in (1.16), is called the fundamental matrix of solutions associated with linear system (1.15). Let $S \subseteq C(I; \mathbb{R}^n)$ be the real linear space consisting of all solutions verifying (1.15). The following properties of the fundamental matrix $\{C(x; x_0), x \in I\}$ are obtained by straight computation.

1.2.1 Properties

Lemma 1.2.1. *Let $x_0 \in I$ be fixed and consider the fundamental matrix $\{C(x; x_0), x \in I\}$ defined in(1.16)*

$$Let \{z(x) : x \in I\} \textit{ be an arbitrary solution of } (1.15) \tag{1.17}$$

Then

$$z(x) = C(x; x_0)z_0, x \in I, where\, z(x_0) = z_0$$

$$\{C(x; x_0), x \in I\} \textit{ is continuously differentiable} \tag{1.18}$$

$$\begin{cases} \frac{dC(x;x_0)}{dx} = & A(x)C(x; x_0), x \in I \\ C(x_0; x_0) = & I_n \textit{ (unit matrix)} \end{cases} \tag{1.19}$$

$$det\, C(x; x_0) \neq 0, \forall x \in I \textit{ and } D(x; x_0) = [C(x; x_0)]^{-1} \textit{ satisfies} \tag{1.20}$$

$$\begin{cases} \frac{dD(x;x_0)}{dx} = & -D(x; x_0)A(x), x \in I \\ D(x_0; x_0) = & I_n \textit{ (unit matrix)} \end{cases} \tag{1.21}$$

$$dim\, S = n \tag{1.22}$$

Proof. For $z(.) \in S$, let $z_0 = z(x_0) \in \mathbb{R}^n$ be column vector and consider $\widehat{z}(x) = C(x;x_0)z_0, x \in I$. Notice that each column of $C(x;x_0)$ is a solution of (1.15). As far as S is a linear space then $\{\widehat{z}(x) : x \in I\}$ is a solution of (1.15) fulfilling the Cauchy condition

$$\widehat{z}(x_0) = C(x_0;x_0)z_0$$

Therefore

$$\{z(x) : x \in I\}\text{and}\{\widehat{z}(x) : x \in I\}$$

are solutions for the linear system (1.15) satisfying the same Cauchy condition $z(x_0) = z_0 = \widehat{z}(x_0)$ and using the uniqueness of the Cauchy problem solution (see Theorem 1.1.2) we get

$$z(x) = \widehat{z}(x) = C(x;x_0)z_0\text{for any}x \in I$$

The conclusion (1.17) is proved. To get (1.19) we notice that each component $\{z_i(x) : x \in I\}$ of $\{C(x;x_0), x \in I\}$, satisfies (1.15) and it shows directly that

$$\frac{dC(x;x_0)}{dx} = \| \frac{dz_1}{dx} ... \frac{dz_n}{dx} \| = A(x)C(x;x_0), x \in I \tag{1.23}$$

and

$$C(x_0;x_0) = \| e_1...e_n \| = I_n \tag{1.24}$$

The property (1.20) can be proved by contradiction. Assuming that $C(x^*;x_0)z_0 = 0$ for some $x^* \in I$ and $z_0 \in \mathbb{R}^n, z_0 \neq 0$, we get that the solution $z^*(x) = C(x;x_0)z_0, x \in I$ of (1.15) satisfies $z^*(x^*) = 0$ which implies (see uniqueness) $z^*(x) = 0, \forall x \in I$ contradicting $z^*(x_0) = z_0 \neq 0$.Therefore $detC(x,x_0) \neq 0$ for any $x \in I$ and denote $D(x;x_0) = [C(x;x_0)]^{-1}$. On the other hand, consider $\{D_1(x;x_0) : x \in I\}$ as the unique solution of the linear matrix system

$$\begin{cases} \frac{dD_1(x;x_0)}{dx} & = -D_1(x;x_0)A(x), x \in I \\ D_1(x_0;x_0) & = I_n(\text{unit matrix}) \end{cases} \tag{1.25}$$

and by straight derivation we get

$$D_1(x;x_0C(x;x_0) = I_n, \forall x \in I \tag{1.26}$$

The equation (1.25) shows that $D(x;x_0) = [C(x;x_0)]^{-1}, x \in I$. The last property (1.22) is a direct consequence of (1.17) and (1.20). Using (1.17), we see easily that $\dim S \leqslant n$ and from (1.20) we obtain that $\{z_1(x), ...z_n(x) : x \in I\}$ are n linearly independent solutions of (1.15). The proof is complete. □

Any basis of S will be called a fundamental system of solutions satisfying (1.15) and we shall conclude this section recalling Liouville'theorem.

Theorem 1.2.2. *Let the $(n \times n)$ continuous matrix $A(x) = [a_{ij}(x)]_{i,j}$ be given and consider n solutions $\{\widehat{z}_1(x), ..., \widehat{z}_n(x) : x \in I\}$ satisfying the linear system* (1.15).

Then $\{W(x) : x \in I\}$ is the solution of the following linear scalar equation

$$\frac{dW(x)}{dx} = (TrA(x))W(x), x \in I\ ,\ W(x) = det \parallel \widehat{z}_1(x), ..., \widehat{z}_n(x) \parallel \tag{1.27}$$

where

$$TrA(x) = \sum_{i=1}^{n} a_{ii}(x)\ \ and\ \ W(x) = W(x_0)exp \int_{x_0}^{x} [TrA(t)]dt, x \in I$$

for some fixed $x_0 \in I$.

Proof. By definition $\frac{d\tilde{z}_i(x)}{dx} = A(x)\tilde{z}_i(x), x \in I, i \in \{1, ..., n\}$ and the matrix $Z(x) = \parallel \tilde{z}_1(x) ... \tilde{z}_n(x) \parallel$ satisfies

$$\frac{dZ(x)}{dx} = A(x)Z(x), x \in I \tag{1.28}$$

Rewrite $Z(x)$ using row vectors

$$Z(x) = \begin{pmatrix} \varphi_1(x) \\ . \\ . \\ . \\ \varphi_n(x) \end{pmatrix}, \alpha \in I \tag{1.29}$$

and from (1.27) we get easily

$$\frac{d\varphi_i(x)}{dx} = \sum_{j=1}^{n} a_{ij}(x)\varphi_j(x), i \in \{1, ..., n\}, x \in I \tag{1.30}$$

where $(a_{i1}(x), ...a_{in}(x))$ stands for the row "i" of the involved matrix $A(x) = (a_{ij}(x))_{i,j \in \{1,...,n\}}$.. On the other hand, the standard rule of derivation for a $\det Z(x) = W(x)$ gives us

$$\frac{dW(x)}{dx} = \sum_{i=1}^{n} W_i(x), \ \ \text{where } W_i(x) = \det \begin{pmatrix} \varphi_1(x) \\ . \\ . \\ . \\ \varphi_{i-1}(x) \\ \frac{d\varphi_i(x)}{dx} \\ \varphi_{i+1}(x) \\ . \\ . \\ . \\ \varphi_n(x) \end{pmatrix} \tag{1.31}$$

and using (1.29) we obtain

$$W_i(x) = a_{ii}(x)W(x)\,,\ i \in \{1, ..., n\} \tag{1.32}$$

Combining (1.30) and (1.32) we get the scalar equation

$$\frac{dW(x)}{dx} = (\sum_{i=1}^{n} a_{ii}(x))W(x), x \in I \tag{1.33}$$

which is the conclusion of theorem.The proof is complete. □

We shall conclude by recalling the constant variation formula used for integral representation of a solution satisfying a linear system.

1.2.2 Constant Variation Formula

Theorem 1.2.3. *(Constant variation formula) We are given continuous mappings*

$$A(x) : I \to M_{n\times n} \text{ and } b(x) : I \to \mathbb{R}^n$$

Let $\{\mathbb{C}(x; x_0), x \in I\}$ *be the fundamental matrix of solutions associated with the linear homogenous system*

$$\frac{dz}{dx} = A(x)z,\ x \in I,\ z \in \mathbb{R}^n \tag{1.34}$$

Let $\{z(x), x \in I\}$*be the unique solution of the linear system with a Cauchy condition* $z(x_0) = z_0$

$$\begin{cases} \frac{dz}{dx} = & A(x)z + b(x), x \in I \\ z(x_0) = & z_0 \end{cases} \tag{1.35}$$

Then

$$z(x) = \mathcal{C}(x; x_0)[z_0 + \int_{x_0}^{x} \mathcal{C}^{-1}(t; x_0)b(t)dt], x \in I \tag{1.36}$$

Proof. By definition $\{z(x), x \in I\}$ fulfilling (1.36) is a solution of the linear system (1.35) provided a straight derivation calculus is used. In addition using $z(x) = z_0$ and uniqueness of the Cauchy problem solution (see Theorem 1.1.2) we get the conclusion. □

Remark 1.2.4. *The constant variation formula expressed in* (1.36) *suggest that the general solution of the linear system* (1.35) *can be written as a sum of the general solution* $\mathbb{C}(x; x_0)z_0, x \in I$ *fulfilling linear homogeneous system* (1.34) *and a particular*

solution (1.35) *(see* $z_0 = 0$*) given by*

$$\mathbb{C}(x;x_0)\int_{x_0}^{x}\mathcal{C}^{-1}(t;x_0)b(t)dt, x \in I \tag{1.37}$$

Remark 1.2.5. *There is no real obstruction for defining solution of a linear system of integral equation*

$$z(x) = z_0 + \int_a^x [A(t)z(t) + b(t)]dt, x \in [a,b] \subseteq \mathbb{R}, z \in \mathbb{R}^n \tag{1.38}$$

where the matrix $A(x) \in M_{n\times n}$ *and the vector* $b(x) \in \mathbb{R}^n$ *are piecewise continuous mappings of* $x \in [a,b]$ *such that* $A(x)$ *and* $b(x)$ *are continuous functions on* $[x_i, x_{i+1})$ *admitting bounded left limits*

$$A(x_{i+1}-0) = \lim_{x\to x_{i+1}} A(x), b(x_{i+1}-0) = \lim_{x\to x_{i+1}} b(x) \text{for each} i \in \{1,2,...,N-1\} \tag{1.39}$$

Here $a = x_0 < x_1 < ... < x_N = b$ *is an increasing sequence such that* $[a,b] = [x_0, x_N]$. *Starting with an arbitrary Cauchy condition* $z(x_0) = z_0 \in \mathbb{R}^n$ *, we construct the corresponding solution* $z(x), x \in [a,b]$*, as a continuous mapping which is continuously differentiable on each open interval* $(x_i, x_{i+1}), i \in \{1,2,...,N-1\}$ *such that the conclusions of the above given result are preserved. We have to take the case of these equations implying derivation* $\frac{dz(x;x_0,z_0)}{dx}$ *of the solution and to mention that they are valid on each open interval* $(x_i, x_{i+1}), i \in \{1,2,...,N-1\}$ *where the matrix* $A(x)$ *and the vector b(x) are continuous functions. Even more, admitting that the components of the matrix* $\{A(x) : x \in [a,b]\}$ *and vector* $\{b(x) : x \in [a,b]\}$*are complex valued functions then the unique Cauchy problem solution of the linear system* (1.38) *is defined as* $\{z(x) \in \mathbb{C}^n : x \in [a,b], z(a) = z_0 \in \mathbb{C}^n\}$ *satisfying* (1.38) $\forall x \in [a,b]$.

1.3 Exercises and Some Problem Solutions

1.3.1 Linear Constant Coefficients Equations(Fundamental System of Solutions

(P_1). Compute the fundamental matrix of solutions $C(x,0)$ for a linear constant coefficients system

$$\frac{dz(x)}{dx} = Az, x \in \mathbb{R}, z \in \mathbb{R}^n, A \in M_{n\times n} \tag{1.40}$$

Solution Since A is a constant matrix we are looking for the fundamental matrix of solutions $C(x) = C(x,0)$, where $x_0 = 0$ is fixed and as it is mentioned in Lemma

(1.2.1) (see equation (1.18)). We need to solve the following matrix equation

$$\begin{cases} \frac{dC(x)}{dx} = & AC(x), x \in \mathbb{R} \\ C(0) = & I_n \end{cases} \tag{1.41}$$

The computation of $\{C(x) : x \in \mathbb{R}\}$ relies on the fact that the unique matrix solution of (1.40) is given by the following matrix exponential series

$$\exp Ax = I_n + \frac{x}{1!}A + ... + \frac{x^k}{k!}A^k + ... \tag{1.42}$$

where the uniform convergence of the matrix series on compact intervals is a direct consequence of comparing it with a numerical convergent series. By direct derivation we get

$$\begin{cases} \frac{d(\exp Ax)}{dx} = & A + \frac{x}{1!}A^2 + ... + \frac{x^k}{k!}A^{k+1} + ... \ = A(\exp Ax), \forall x \in \mathbb{R} \\ (\exp Ax)_{x=0} = & I_n \end{cases} \tag{1.43}$$

and it shows that $C(x) = \exp Ax, \forall x \in \mathbb{R}$. It suggest the first method of computing C(x) by using partial sum of the series (1.42)
(P_2). Find a fundamental system of solutions(basis of S) for the linear constant coefficients system

$$\frac{dz}{dx} = Az, x \in \mathbb{R}, z \in \mathbb{R}^n, A \in M_{n\times n} \tag{1.44}$$

using the eigenvalues $\lambda \in \sigma(A)$(spectrum of A).
Case I
The characteristic polynomial $det(A-\lambda I_n) = P(\lambda)$ has n real distinct roots $\{\lambda_1, ...\lambda_n\} \subseteq \mathbb{R}, i.e\, \sigma(A) = \{\lambda_1, ...\lambda_n\}, \lambda_i \neq \lambda_j, i \neq j$. Let $v_i \neq 0, v_i \in \mathbb{R}^n$ be an eigenvector corresponding to the eigenvalue $\lambda_i \in \sigma(A)$, i.e

$$Av_i = \lambda_i v_i, i \in \{1, ..., n\} \tag{1.45}$$

By definition ,$\{v_1, .., v_n\} \subseteq \mathbb{R}^n$is a basis in $\mathbb{R}^n$ and define the vector functions

$$\widehat{z}_i(x) = (\exp\lambda_i x)v_i, x \in \mathbb{R}, i \in \{1, ..., n\} \tag{1.46}$$

Each $\{\widehat{z}_i(x) : x \in \mathbb{R}\}$ satisfies (1.43) and $\{\widehat{z}_1(x), ..., \widehat{z}_n(x)\}$ is a basis of the linear space S consisting of all solutions satisfying (1.43). Therefore,any solution$\{z(x) : x \in I\}$ of the system (1.43) can be found as a linear combination of $\widehat{z}_1(.), ..., \widehat{z}_n(.)$ and $\{\alpha_1, ..., \alpha_n\}$ from $z(x) = \sum_{i=i}^n \alpha_i \tilde{z}_i(x), x \in \mathbb{R}$ will be determined by imposing Cauchy condition $z(0) = z_0 \subseteq \mathbb{R}^n$.
Case II
The characteristic polynomial $P(\lambda) = \det(A - \lambda I_n)$ has n complex numbers as roots $\{\lambda_1, ..., \lambda_n\} = \sigma(A), \lambda_i \neq \lambda_j, i \neq j$. As in the real case we define n complex valued solutions satisfying (1.43)

$$\widehat{z}_j(x) = (\exp\lambda_j x)v_j, j \in \{1, ..., n\} \tag{1.47}$$

where $v_j \subseteq \mathbb{R}^n$ is an eigenvector corresponding to a real eigenvalue $\lambda_j \in \sigma(A)$ and $v_j \in \mathbb{C}^n$ is a complex eigenvector corresponding to the complex eigenvalue $\lambda_j \in \sigma(A)$ such that $\bar{v_j} = a_j - ib_j, (v_j = a_j + ib_j)$ is the eigenvector corresponding to the eigen value $\overline{\lambda}_j = \alpha_j - i\beta_j, (\lambda_j = \alpha_j + i\beta_j, \beta_j \neq 0)$. From $\{\widehat{z}_1(x), ..., \widehat{z}_n(x); x \in \mathbb{R}\}$ defined in (1.47) we construct another n real solutions $\{\widehat{z}_1(x), ..., \widehat{z}_n(x); x \in \mathbb{R}\}$ as follows $(n = m + 2k)$. The first m real solutions

$$\widehat{z}_i(x) = (\exp\lambda_i x)v_i, i \in \{1, ..., m\} \tag{1.48}$$

when $\{\lambda_1, ..., \lambda_m\} \subseteq \sigma(A)$ are the real eigenvalues of A and another $2k$ real solutions

$$\begin{cases} \widehat{z}_{m+j}(x) = & \mathrm{Re}\,\widehat{z}_{m+j}(x) = \frac{\widehat{z}_{m+j}(x)+\widehat{z}_{m+j+k}(x)}{2} \\ \widehat{z}_{m+k+j}(x) = & \mathrm{Im}\,\widehat{z}_{m+j}(x) = \frac{\widehat{z}_{m+j}(x)-\widehat{z}_{m+k+j}(x)}{2i} \end{cases} \tag{1.49}$$

for any $j \in \{1, ..., k\}$. Here $\{\lambda_{m+1}, ..., \lambda_{m+2k}\} \subseteq \sigma(A)$ are the complex eigenvalues such that $\lambda_{m+j+k} = \overline{\lambda}_{m+j}$ for any $j \in \{1, ..., k\}$. Since $\{\widehat{z}_1(x), ..., \widehat{z}_n(x); x \in \mathbb{R}\}$ are linearly independent over reals and the linear transformation used in (1.47) and (1.48) is a nonsingular one, we get $\{\widehat{z}_1(x), ..., \widehat{z}_n(x); x \in \mathbb{R}\}$as a basis of S.

Case III: General Case

$\sigma(A) = \{\lambda_1, ...\lambda_d\} \subseteq \mathbb{R}$, where the eigenvalue λ_j has a multiplicity n_j and $n = n_1 + ... + n_d$.In this case the canonical Jordan form of the matrix A is involved which allows to construct a basis of S using an adequate transformation $T : \mathbb{R}^n \to \mathbb{R}^n$. It relies on the factors decomposition of the characteristic polynomial $\det(A - \lambda I_n) = P(\lambda) = (\lambda - \lambda_1)^{n_1}...(\lambda - \lambda_d)^{n_d}$ and using Caylay-Hamilton theorem we get

$$\begin{cases} P(A) = 0(\text{null matrix}) \\ (A - \lambda_1 I_n)^{n_1}...(A - \lambda_d I_n)^{n_d} = 0(\text{null mapping} : \mathbb{C}^n \to \mathbb{C}^n) \end{cases} \tag{1.50}$$

The equations (1.49) are essential for finding a $n \times n$ nonsingular matrix $Q =\| v_1...v_n \|$ such that the linear transformation $z = Qy$ leads us to a similar matrix $B = Q^{-1}AQ(B \sim A)$ for which the corresponding linear system

$$\frac{dy}{dx} = By, B = \mathrm{diag}(B_1, ..., B_d), \dim B_j = n_j, j \in \{1, ..., d\} \tag{1.51}$$

has a fundamental matrix of solution.

$$Y(x) = \exp Bx = \mathrm{diag}(\exp B_1 x, ..., \exp B_d x)$$

Here

$$B_j = \lambda_j I_{n_j} + C_{n_j}\ , \text{and} \begin{pmatrix} 0 & 0 & 0 & . & . & 0 \\ 1 & 0 & 0 & . & . & 0 \\ 0 & 1 & 0 & . & . & o \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & 1 & 0 \end{pmatrix} = C_{n_j}$$

has the nilpotent property $(C_{n_j})^{n_j} = 0$ (null matrix). The general case can be computed in a more attractive way when the linear constant coefficients system comes from n-th order scalar differential equation . In this respect consider a linear n-th

order scalar differential equation

$$y^n(x) + a_1 y^{(n-1)}(x) + ... + a_n y(x) = 0, x \in \mathbb{R} \tag{1.52}$$

Denote $z(x) = (y(x), y^{(1)}(x), ..., y^{(n-1)}(x)) \in \mathbb{R}^n$ and (1.51) can be written as a linear system for the unknown z.

$$\frac{dz}{dx} = A_0 z, z \in \mathbb{R}^n, x \in \mathbb{R} \tag{1.53}$$

where

$$A_0 = \begin{pmatrix} b_1 \\ . \\ . \\ . \\ b_n \end{pmatrix}$$

$$b_1 = (0, 1, 0, ..., 0), ...b_{n-1} = (0, ..., o, 1) \text{and}\, b_n = (-a_n, ..., -a_1)$$

The characteristic polynomial associated with A_0 is given by

$$P_0(\lambda) = det(A_0 - \lambda I_n) = \lambda^n + a_1\lambda^{n-1} + ... + a_{n-1}\lambda + a_n \tag{1.54}$$

where $a_i \in \mathbb{R}, i \in \{1, ..., n\}$ are given in (1.52). In this case, a fundamental system of solutions (basis) $\{y_1(x), ..., y_n(x) : x \in \mathbb{R}\}$ for (1.51) determines a basis $\{z_1(x), ..., z_n(x) : x \in \mathbb{R}\})$ for (1.52), where

$$z_i(x) = column\{y_i(x), y_i^{(1)}(x)..., y_i^{(n-1)}(x)\}, i \in \{1, ..., n\}$$

In addition, a basis $\{y_1(x), y_2(x)..., y_n(x)\}$ for the scalar equation (1.51) is given by $F = \bigcup_{j=1}^d F_j$ where

$$F_j = \{(\exp\lambda_j x), x(\exp\lambda_j x), ..., x^{n_j-1}(\exp\lambda_j x)\} \tag{1.55}$$

if the real $\lambda_j \in \sigma(A_0)$ has the multiplicity degree n_j, and

$$F_j = Fj(cos) \bigcup F_j(sin), \text{ if the complex}\, \lambda_j \in \sigma(A_0)\, ,\, \lambda_j = \alpha_j + i\beta_j \tag{1.56}$$

has the multiplicity degree n_j, where

$$F_j(cos) = \{(\exp\alpha_j x), x(\exp\alpha_j x), ..., x^{n_j-1}(\exp\alpha_j x)\}(cos\beta_j x), x \in \mathbb{R}$$

$$F_j(sin) = \{(\exp\alpha_j x), x(\exp\alpha_j x), ..., x^{n_j-1}(\exp\alpha_j x)\}(sin\beta_j x), x \in \mathbb{R}$$

1.3.2 Some Stability Problems and Their Solution

(P_1). $\widehat{z}(x;z_0) = [\exp Ax]z_0, x \in [0,\infty)$, be the solution of the

$$\frac{dz}{dx} = Az, x \in [0,\infty), z(0) = z_0 \in \mathbb{R}^n \tag{1.57}$$

We say that $\{\widehat{z}(x,z_0) : x \geqslant 0\}$is exponentially stable if

$$\mid \widehat{z}(x,z_0) \mid \leqslant \mid z_0 \mid (\exp - \gamma x), \forall x \in [0,\infty), z_0 \in \mathbb{R}^n \tag{1.58}$$

where the constant $\gamma > 0$ does not depend on z_0. Assume that

$$\sigma(A + A^t) = \{\lambda_1, ..., \lambda_d\} \,\text{satisfies}\, \lambda_i < 0 \,\text{for any}\, i \in \{1, ..., d\} \tag{1.59}$$

Then $\{\widehat{z}(x,z_0) : x \in [0.\infty)\}$is exponentially stable
Solution of P_1
For each $z_0 \in \mathbb{R}^n$, the corresponding Cauchy problem solution $\widehat{z}(x,z_0) = (\exp Ax)$ $z_0, x \geqslant 0$, satisfies (1.56) and in addition,the scalar function $\{\varphi(x) =\mid \widehat{z}(x,z_0) \mid^2, x \geqslant 0\}$ fulfils the following differential inequality

$$\frac{d\varphi(x)}{dx} = \langle (A + A^t)\widehat{z}(x,z_0), \widehat{z}(x,z_0)\rangle \leqslant 2w\varphi(x), \forall x \geqslant 0 \tag{1.60}$$

where

$$2w = \max\{\lambda_1, ..., \lambda_d\} < o$$

provided the condition (1.58) is assumed. A simple explanation of this statement comes from the diagonal representation of the symmetric matrix $(A + A^t)$ when an orthogonal transformation $T : \mathbb{R}^n \to \mathbb{R}^n, z = Ty(T = T^{-1})$is performed.We get

$$\langle (A + A^t)\widehat{z}(x,z_0), \widehat{z}(x,z_0)\rangle = \langle [T^{-1}(A + A^t)T]\widehat{y}(x;z_0), \widehat{y}(x;z_0)\rangle \tag{1.61}$$

where

$$T^{-1}(A + A^t)T = \mathrm{diag}(\nu_1, ..., \nu_n)$$

and

$$\nu_i \in \sigma(A + A^t) \;\; \text{for each}\; \{1, ..., n\}$$

Using (1.60)we see easily that

$$\langle (A + A^t)\widehat{z}(x,z_0), \widehat{z}(x,z_0)\rangle \leqslant [\max\{\lambda_1, ...\lambda_d\}] \mid \widehat{z}(x,z_0) \mid^2 = 2w\varphi(x) \tag{1.62}$$

where

$$2w = \max\{\lambda_1, ..., \lambda_d\} < o$$

and

$$\varphi(x) =\mid \widehat{z}(x;,z_0) \mid^2 = \langle T\widehat{y}(x;z_0), T\widehat{y}(x;z_0)\rangle = \langle \widehat{y}(x;z_0), T^tT\widehat{y}(x;z_0)\rangle =\mid \widehat{y}(x;z_0) \mid^2$$

are used. The inequality (1.61) shows that (1.59) is valid and denoting

$$0 \geqslant \beta(x) = \frac{d\varphi(x)}{dx} - 2w\varphi(x), x \geqslant 0$$

we rewrite (1.59) as a scalar differential equation

$$\begin{cases} \frac{d\varphi(x)}{dx} = & 2w\varphi(x) + \beta(x), x \geqslant 0 \\ \varphi(0) = & \mid z_0 \mid \end{cases} \tag{1.63}$$

where $\beta(x) \leqslant 0, x \in [0, \infty)$, is a continuous function.The unique solution of (1.62) can be represented by

$$\varphi(x) = [\exp 2wx][\mid z_0 \mid^2 + \int_0^x (\exp - 2wt)\beta(t)dt] \tag{1.64}$$

and using $(\exp - 2wt)\beta(x) \leqslant 0$ for any $t \geqslant 0$ we get

$$\begin{cases} \varphi(x) \leqslant & [\exp 2wx] \mid z_0 \mid^2 \\ \mid \widehat{z}(x, z_0) = & [\varphi(x)]^{\frac{1}{2}} \leqslant [\exp wx] \mid z_0 \mid \end{cases} \tag{1.65}$$

for any $x \geqslant 0$. It shows that the exponential stability expressed in (1.57) is valid when the condition (1.58) is assumed.
(P_2) (Lyapunov exponent associated with linear system and piecewise continuous solutions) Let $\{y(t, x) \in \mathbb{R}^n : t \geqslant 0\}$ be the unique solution of the following linear system of differential equations

$$\begin{cases} \frac{dy(t)}{dt} = & f(y(t), \widehat{\sigma}(t)), t \geqslant 0, y(t) \in \mathbb{R}^n \\ y(0) = & x \in \mathbb{R}^n \end{cases} \tag{1.66}$$

where the vector field $f(y, \sigma) : \mathbb{R}^n \times \sum \to \mathbb{R}^n$ is a continuous mapping of $(y, \sigma) \in \mathbb{R}^n \times \sum$ and linear with respect to $y \in \mathbb{R}^n$

$$f(y, \sigma) = A(\sigma)y + a(\sigma), A(\sigma) \in M_{n\times n}, a(\sigma) \in \mathbb{R}^n\,. \tag{1.67}$$

Here $\widehat{\sigma}(t) : [0, \infty) \to \sum$(bounded set) $\subseteq \mathbb{R}^d$ is an arbitrary piecewise continuous function satisfying

$$\widehat{\sigma(t)} = \widehat{\sigma(\widehat{t_k})}, t \in [\widehat{t_k}, \widehat{t_{k+1}}), k \geqslant 0 \tag{1.68}$$

where $0 = \widehat{t_0} \leqslant \widehat{t_1} ... \leqslant \widehat{t_k}$ is an increasing sequence with $\underset{k\to\infty}{lim}\ \widehat{t_k} = \infty$. The analysis will be done around a piecewise constant trajectory $\widehat{y}(t) : [0, \infty) \to Y$(bounded set) $\subseteq R^n$ such that

$$\widehat{\lambda}(t) = (\widehat{y}(t), \widehat{\sigma}(t)) : [o, \infty) \to Y \times \sum = \Lambda \subseteq \mathbb{R}^n \times \mathbb{R}^d \tag{1.69}$$

satisfies $\widehat{\lambda}(t) = \widehat{\lambda}(\widehat{t_k}), t \in [\widehat{t_k}, \widehat{t_{k+1}}), k \geqslant 0$, where the increasing sequence $\{\widehat{t_k}\}_{k\geqslant 0}$ is fixed in (1.68). Define a linear vector field $g(z, \lambda)$ by

$$g(z, \lambda) = f(z + \nu; \sigma) = A(\sigma)z + f(\lambda), \lambda = (\nu, \sigma) \in \Lambda, z \in \mathbb{R}^n \text{ where } f(y; \sigma) \tag{1.70}$$

is given in (1.66). Let the continuous mapping $\{z(t,x) : t \geqslant 0\}$ be the unique solution of the following differential equation

$$\begin{cases} \frac{dz(t)}{dt} = & g(z(t);\widehat{\lambda}(t)), t \geqslant 0 \\ z(0) & = x \end{cases} \tag{1.71}$$

where the piecewise continuous function $\{\widehat{\lambda}(t), t \geqslant 0\}$ is given in (1.68). We may and do associate the following piecewise continuous mapping $\{z(t,x) : t \geqslant 0\}$ satisfying the following linear system with jumps

$$\begin{cases} \frac{d\widehat{z}(t)}{dt} = & f(\widehat{z}(t),\widehat{\sigma}(t)), t \in [\widehat{t}_k, \widehat{t}_{k+1}), \widehat{z}(t) \in \mathbb{R}^n \\ \widehat{z}(\widehat{t}_k) = & z(\widehat{t}_k, x) + \widehat{y}(t_k), k \geqslant 0 \end{cases} \tag{1.72}$$

where $\{z(t,x) : t \geqslant 0\}$is the continuous mapping satisfying (1.70). It is easily seen that the piecewise continuous mapping $\{\widehat{z}(t,x) : t \geqslant 0\}$ can be decomposed as follows

$$\widehat{z}(t,x) = z(t,x) + \widehat{y}(t,x), t \geqslant 0, x \in \mathbb{R}^n \tag{1.73}$$

and the asymptotic behavior
$\underset{t\to\infty}{lim}\,\widehat{z}(t,x) = \underset{t\to\infty}{lim}\,\widehat{y}(t), x \in \mathbb{R}^n$, is valid provided $\underset{t\to\infty}{lim}\,\widehat{y}(t)$ exists and

$$\underset{t\to\infty}{lim} \mid z(t,x) \mid^2 = 0 \tag{1.74}$$

for $x \in \mathbb{R}^n$. We say that$\{z(t,x) : t \geqslant 0\}$ satisfying (1.70) is asymptotically stable if (1.73) is valid.

Definition 1.3.1. *A constant $\gamma < 0$ is a Lypounov exponent for $\{z(t,x) : t \geqslant 0\}$ if $\{\widehat{z}(t,x) = (exp\gamma t)z(t,x) : t \geqslant 0\}$is asymptotically stable.*

Remark 1.3.2. *Notice that if $\{\widehat{z}(t,x) : t \geqslant 0\}$ satisfies* (1.70) *then $\{z_\gamma(t,x) = (exp\gamma t)z(t,x) : t \geqslant 0\}$, satisfies the following augmented linear system*

$$\begin{cases} \frac{dz_\gamma(t)}{dt} = & \gamma z_\gamma(t) + (exp\gamma t)g(z(t),\widehat{\lambda}(t)), t \geqslant 0 \\ z_\gamma(0) = & x \end{cases} \tag{1.75}$$

In addition, the piecewise continuous mapping

$$\widehat{z}_\gamma(t,x) = z_\gamma(t,x) + \widehat{y}(t), t \geqslant 0 \tag{1.76}$$

satisfies the following system with jumps

$$\begin{cases} \frac{d\widehat{z}_\gamma(t)}{dt} = & \gamma[\widehat{z}_\gamma(t) - \widehat{y(t)}] + (exp\gamma t)f(\widehat{z}_\gamma(t),\widehat{\lambda}(t)), t \geqslant 0 \\ \widehat{z}_\gamma(\widehat{t}_k) = & z_\gamma(\widehat{t}_k, x) + \widehat{y}(\widehat{t}_k)\,. \end{cases} \tag{1.77}$$

It shows that the Lyapunov exponent $\gamma < 0$ found for the continuous mapping $\{z(t,x) : t \geqslant 0\}$ satisfying (1.70) *gives the answer for the following asymptotic behavior associated with $\{\widehat{z}(t,x)\}$*

$$\underset{t\to\infty}{lim} \mid \widehat{z}_\gamma(t,x) - \widehat{y}(t) \mid^2 = \underset{t\to\infty}{lim} \mid z_\gamma(t,x) \mid = 0\,, for\ each\, x \in \mathbb{R}^n \tag{1.78}$$

and the analysis will be focussed on getting Lyapunov exponents for the continuous mapping $\{z(t,x) : t \geqslant 0\}$

A description of the Lyapunov exponent associated with the continuous mapping $\{z(t,x) : t \geqslant 0\}$ can be associated using the corresponding integral equation satisfied by a scalar continuous function

$$h_\gamma(t,x) = (\exp 2\gamma t)h(z(t,x)), t \geqslant 0, \text{ where } h(z) =\mid z \mid^2 \tag{1.79}$$

In this respect,applying standard rule of derivation we get

$$\begin{cases} \frac{d\widehat{z}_\gamma(t,x)}{dt} = & (\exp 2\gamma t)[2\gamma h + L_g(h)](\widehat{z}_\gamma(t,x), \widehat{\lambda}(t), t \geqslant 0 \\ h_\gamma(0,x) = & \mid x \mid^2 \end{cases} \tag{1.80}$$

where the first order differential operator $L_g : P_2(z,\lambda) \to P_2(z,\lambda)$ is given by

$$L_g(\varphi)(z,\lambda) =< \partial_z \varphi(z,\lambda)), g(z,\lambda) > \tag{1.81}$$

Here $P_2(z,\lambda)$ consist of all polynomial scalar functions of second degree with respect to the variables $z = (z_1, ..., z_n)$ and with continuous coefficients as functions of $\lambda \in \Lambda$. In particular, for $h \in P_2(z,\lambda)$, $h(z) =\mid z \mid^2$, we obtain

$$\begin{aligned} L_g(\varphi)(z,\lambda) &= < \partial_z h(z), g(z,\lambda) >=< [A(\sigma) + A^*(\sigma)z, z] > +2 < f(\lambda), z > \\ &= < B(\sigma)z, z > +2 < f(\lambda), z > \end{aligned} \tag{1.82}$$

where the matrix $B(\sigma) = A(\sigma) + A^*(\sigma)$is symmetric for each $\sigma \in \sum \subseteq \mathbb{R}^d$ and for $f(\lambda) = A(\sigma)v + a(\sigma), \lambda = (v,\sigma) \in \Lambda = Y \times \sum$, given in (1.66). Rewrite (1.79) as follows (see(1.71))

$$\begin{aligned} \frac{dh_\gamma(t,x)}{dt} &= - \mid \gamma \mid h_\gamma(t,x) - (\exp 2\gamma t) < [\mid \gamma \mid I_n - B(\widehat{\sigma}(t))]z(t,x) > \\ &+ \; 2(\exp 2\gamma t) < f(\widehat{\lambda}(t)), z(t,x) >, t \geqslant 0\,. \end{aligned} \tag{1.83}$$

Regarding the symmetric matrix

$$Q_\gamma(\sigma) =\mid \gamma \mid I_n - B(\sigma), \sigma \in \sum \subseteq \mathbb{R}^d$$

we notice that it can be defined as positively defined matrix uniformly with respect to $\sigma \in \sum$

$$< Q_\gamma(\sigma)z, z >\geqslant c \mid z \mid^2, \forall z \in \mathbb{R}^n, \sigma \in \sum, \text{for some } c \geqslant 0 \tag{1.84}$$

provided, $\mid \gamma \mid\geqslant\parallel B \parallel$, where

$$\parallel B \parallel= \sup_{\sigma \in \sum} \parallel B(\sigma) \parallel$$

In this respect, let $T(\sigma) : \mathbb{R}^n \to \mathbb{R}^n$ be an orthogonal matrix $(T^*(\sigma) = T^{-1}(\sigma))$ such

that

$$\begin{cases} T^{-1}(\sigma)B(\sigma)T(\sigma) = & \mathrm{diag}(\nu_1(\sigma), ..., \nu_n(\sigma)) = \Gamma_\gamma(\sigma) \\ B(\sigma)e_j(\sigma) = & \nu_j(\sigma)e_j(\sigma), j \in \{1, 2, ..., n\} \end{cases} \tag{1.85}$$

where $T(\sigma) = \| e_1(\sigma), ..., e_n(\sigma) \|$. On the other hand ,using the same matrix $T(\sigma)$ we get

$$T^{-1}(\sigma)Q_\gamma(\sigma)T(\sigma) = \mathrm{diag}(\nu_1^\gamma(\sigma), ..., \nu_n^\gamma(\sigma)) = \Gamma_\gamma(\sigma) \tag{1.86}$$

where $\nu_i^\gamma(\gamma) = \mid \gamma \mid -\nu_i(\sigma) \geqslant c > 0$ for any $\sigma \in \sum, i \in \{1, ..., n\}$, provided $\mid \gamma \mid > \| B \|$ and we get

$$\mid \gamma \mid > \| B \| \geqslant \sup_{\sigma \in \sum} \| B(\sigma) \| \geqslant \| B(\sigma)e_j(\sigma) \| = \sup_{\sigma \in \sum} \mid \nu_j(\sigma) \mid, \forall j \in \{1, ..., n\} \tag{1.87}$$

Using (1.85), it makes sense to consider the square root of the positively defined matrix $Q_\gamma(\sigma)$

$$\sqrt{[Q_\gamma(\sigma)]} = T(\sigma)[\Gamma_\gamma(\sigma)]^{\frac{1}{2}}.T^{-1}(\sigma) = P_\gamma(\sigma), \sigma \in \sum \subseteq \mathbb{R}^d \tag{1.88}$$

and rewrite $< Q_\gamma(\sigma)z, z > -2 < f(\lambda), z >= \varphi_\gamma(z, \lambda)$ as follows

$$\varphi(z, \lambda) = \mid P_\gamma(\sigma)z - R_\gamma(\sigma)f(\lambda) \mid^2 - \mid R_\gamma(\sigma)f(\lambda) \mid^2 \tag{1.89}$$

where

$$R_\gamma(\sigma) = \sqrt{Q_\gamma^{-1}(\sigma)} = T(\sigma)[\Gamma_\gamma(\sigma)]^{-1/2}T^{-1}(\sigma) \ .$$

Using (1.88) we get the following differential equation

$$\begin{aligned} \frac{dh_\gamma(t;x)}{dt} &= -\mid \gamma \mid h_\gamma(t;x) - (\exp 2\gamma t) \mid P_\gamma(\widehat{\sigma}(t)z(t;x) - R_\gamma(\widehat{\sigma}, t)f(\widehat{\lambda}(t))) \mid^2 \\ &+ (\exp 2\gamma t) \mid R_\gamma(\widehat{\sigma}(t))f(\widehat{\lambda})(t) \mid^2, t \geqslant 0 \end{aligned} \tag{1.90}$$

The integral representation of the solution $\{h_\gamma(t;x) : t \geqslant 0\}$ fulfilling (1.90) leads us directly to

$$\begin{aligned} h_\gamma(t;x) &= (\exp\gamma t)[\mid x \mid^2 + \int_0^t (\exp\gamma s) \mid R_\gamma(\widehat{\sigma})(s)f(\widehat{\lambda}(s)) \mid^2 ds] \qquad (1.91) \\ &- (\exp\gamma t) \int_0^t (\exp\gamma s) \mid P_\gamma(\widehat{\sigma}(t)z(s;x) - R_\gamma(\widehat{\sigma})(s)f(\widehat{\lambda}(s))) \mid^2 ds \end{aligned}$$

for any $t \geqslant 0 and x \in \mathbb{R}^n$. As far as $\{R_\gamma(\widehat{\sigma})(t)f(\widehat{\lambda}(t)) : t \geqslant 0\}$ is a continuous and bounded function on $[0, \infty)$ we obtain that

$$\int_0^t (\exp\gamma s) \mid R_\gamma(\widehat{\sigma})(s)f(\widehat{\lambda}(s)) \mid^2 ds \leqslant C_\gamma, \forall t \geqslant 0 \tag{1.92}$$

and

$$h_\gamma(t, x) \leqslant (\exp\gamma t)[\mid x \mid^2 + C_\gamma], \forall t \geqslant 0 \tag{1.93}$$

where C_γ is a constant. In conclusion, for each $\gamma < 0, \mid \gamma \mid > \sup_{\sigma\in\sum} \mid A(\sigma) + A^*(\sigma) \mid$, we obtain

$$\lim_{t\to\infty} \mid z_\gamma(t,x) \mid^2 = \lim_{t\to\infty} h_\gamma(t,x) = 0$$

for each $x \in \mathbb{R}$. The above given computations can be stated as

Theorem 1.3.3. *Let the vector field* $f(y,\sigma) : \mathbb{R}^n \times \sum \to \mathbb{R}^n$ *be given such that* (1.66) *is satisfied. Then any* $\gamma < 0$ *satisfying* $\mid \gamma \mid > \sup_{\sigma\in\sum} \mid A(\sigma) + A(\sigma^*) \mid$ *is a Lyapunov exponent for the continuous mapping* $\{z(t,x) : t \geqslant 0\}$ *verifying* (1.70)*, where* $\widehat{\lambda}(t) = (\widehat{y}(t), \widehat{\sigma}(t)) : [0,\infty) \to \Lambda$ *is fixed arbitrarily. In addition, let* $\{\widehat{z}(t,x) : t \geqslant 0\}$ *be the piecewise continuous solution fulfilling the corresponding system with jumps* (1.76)*. Then* $\lim_{t\to\infty} \mid \widehat{z}_\gamma(t,x) - \widehat{y}(t) \mid = \lim_{t\to\infty} \mid z_\gamma(t,x) \mid = 0$ *for each* $x \in \mathbb{R}^n$

Remark 1.3.4. *The result stated in Theorem 1.3.3 make use of some bounds* $\sup_{\sigma\in\Sigma} \mid A(\sigma) + A(\sigma^*) \mid = \parallel B \parallel$ *associated with the unknown matrix* $A(\sigma), \sigma \in \Sigma$ *(bounded set)* $\subseteq \mathbb{R}^d$*. In the particular case* $\Sigma = \{\sigma_1, ..., \sigma_m\}$*, we get a finite set of matrices* $A(\sigma_1), ...A(\sigma_m)$ *for which* $\parallel B \parallel = max\{\mid A_1 + A_1^* \mid, ..., \mid A_m + A_m^* \mid\}$ *where* $A_i = A(\sigma_i)$*.*
An estimate of the the Theorem 1.3.3 stating that $lim_{t\to\infty}(exp\gamma t) \mid z(t,x) \mid = 0$ *provided* $\mid \gamma \mid = -\gamma > \parallel B \parallel$ *and in addition* $z(t,x), t \geqslant 0$ *can be measured as continuous solution of the system* (1.70)*. This information can be used for estimating the bounds* $\parallel B \parallel$*and as far as we get* $lim_{t\to\infty}(exp\widehat{\gamma t}) \mid z(t,x) \mid = 0$ *for some* $\widehat{\gamma} < 0$*. We may predict that* $\parallel B \parallel < \mid \widehat{\gamma} \mid$*. On the other hand ,if we are able to measure only some projections*
$\nu_i(t,x) = < b_i, z(t,x) >, i \in \{1,2,..,m\}$ *of the solution* $z(t,x) : t \geqslant 0$ *satisfying* (1.70) *then* $\nu(t,x) = (\nu_1(t,x), ...\nu_m(t,x))$ *fulfils the following linear equation*

$$\begin{cases} \frac{d\nu(t,x)}{dt} = & D(\widehat{\gamma}(t))\nu(t,x) + d(\widehat{\lambda}(t)), t \geqslant 0 \\ \nu(0,x) = & \nu^0(x) = (b_1, x >, ..., b_m, x >) \end{cases} \tag{1.94}$$

where $D(\sigma)$ *is an* $(m \times m)$ *continuous matrix and*
$d(\sigma) = column(< b_1, f(\lambda) >, ..., < b_m, f(\lambda) >)$
Here we have assumed that $A^*(\sigma)[b_1, ..., b_m] = [b_1, ..., b_m]D^*(\sigma)$ *and* (1.3.3) *gets the corresponding version when* (1.93) *replaces the original system* (1.70)

1.4 Nonlinear Systems of Differential Equations

Let $f(x, \lambda, y) : I \times \Lambda \times G \to \mathbb{R}^n$be a continuous function,where $I(interval) \subseteq \mathbb{R}$ and $\Lambda \subseteq \mathbb{R}^m$ are some open sets. Consider a system of differential equations(normal form)

$$\frac{dy}{dx} = f(x, \lambda, y) \tag{1.95}$$

By a solution of (1.94) we mean a continuous function $y(x,\lambda) : J \times \Sigma \to G$ which is continuously derivable with respect to $x \in J$ such that

$$\frac{dy(x,\lambda)}{dx} = f(x,\lambda,y(x,\lambda)), \forall x \in J, \lambda \in \Sigma$$

where $J \subseteq I$ is an interval and $\Sigma \subseteq \Lambda$ is a compact set. The Cauchy problem for the nonlinear system (1.94) $C.P(f;x_0,y_0)$, has the meaning that we must determine a solution of (1)$y(x,\lambda) : J \times \Sigma \to G$ which satisfies $y(x_0,\lambda) = y_0, \lambda \in \Sigma$,where $x_0 \in I$and $y_0 \in G$ are fixed. To get a unique $C.P(f;x_0,y_0)$ solution we need to replace the continuity property of f by a Lipschitz condition with respect to $y \in G$

Definition 1.4.1. *We say that a continuous function $f(x,\lambda,y) : I \times \Lambda \times G \to \mathbb{R}^n$ is locally Lipschitz continuous with respect to $y \in G$ if for each compact set $W = J \times \sum \times K \subseteq I \times \Lambda \times G$ there exists a constant $L(W) > 0$ such that$| f(x,\lambda,y'') - f(x,\lambda,y') \leqslant L \mid y'' - y' \mid, \forall y', y'' \in K, x \in J, \lambda \in \Lambda$*

Definition 1.4.2. *We say that a $C.P(f;x_0,y_0)$ solution $\{y(x;\lambda) : x \in J, \lambda \in \sum\}$is unique if for any other solution $\{y_1(x;\lambda) : x \in J_1, \lambda \in \sum_1\}$ of (1.95) which verifies $y_1(x_0) = y_0$,we get $y(x,\lambda) = y_1(x,\lambda), \forall (x,\lambda) \in (J \times \sum) \bigcap (J_1 \times \sum_1)$.Denote $B(y_0,b) \subseteq \mathbb{R}_n$ the ball centered at $y_0 \in \mathbb{R}^n$ whose radius is $b > 0$*

Remark 1.4.3. *By a straight computation we get that if the right hand side of (1.94) is continuously differentiable function with respect to $y \in G$, i.e*

$$\frac{\partial f}{\partial y_i}(x,\lambda,y) : I \times \Lambda \times G \to \mathbb{R}^n, i \in \{1,...,n\}$$

are continuous functions and G is a convex domain then f is locally Lipschitz with respect to $y \in G$

1.4.1 Existence and Uniqueness of C.P(f, x_0, y_0)

Theorem 1.4.4. *(Cauchy Lipschitz) Let the continuous function $f(x;\lambda,y) : I \times \Lambda \times G \to \mathbb{R}^n$ be locally Lipschitz continuous with respect to $y \in G$, where $I \subseteq \mathbb{R}, \Lambda \subseteq \mathbb{R}^m, G \subseteq \mathbb{R}^n$ are open sets.For some $x_0 \in I, y_0 \in G$ and $\Sigma(compact) \subseteq \Lambda$ fixed, we take $a, b > 0$ such that $I_a(x_0) = [x_0 - a, x_0 + a] \subseteq I$and $B(y_0,b) \subseteq G$. Let $M = max\{| f(x,\lambda,y) |: x \in I_a(x_0), \lambda \in \Sigma, y \in B(y_0,b)\}$. Then there exist $\alpha > 0, \alpha = min(a, \frac{b}{M})$ and a unique $C.P\ (f;x_0,y_0)$ solution $y(x,\lambda) : I_\alpha(x_0) \times \Sigma \to B(y_0,b)$ of (1.95).*

Proof. We associate the corresponding integral equation (as in the linear case)

$$y(x,\lambda) = y_0 + \int_{x_0}^{x} f(t,\lambda,y(t,\lambda))dt, x \in I_a(x_0), \lambda \in \Sigma \tag{1.96}$$

where $I_a(x_0) \subseteq I$ and $\Sigma(compact) \subseteq \Lambda$ are fixed. By a direct inspection, we see that the two systems (1.94) and (1.95) are equivalent using their solutions and the

existence of solution for (1.94) with $y(x_0) = y_0$ will be obtained proving that (1.95) has a solution.In this respect, a sequence of continuous functions $\{y_k(x, \lambda) : (x, \lambda) \in I_\alpha(t_0) \times \Sigma\}_{k \geqslant 0}$ is constructed such that

$$y_0(x, \lambda) = y_0, y_{k+1}(x, \lambda) = y_0 + \int_{x_0}^{x} f(t, \lambda, y_k(t, \lambda))dt, k \geqslant 0 \tag{1.97}$$

Consider $\alpha = \min(a, \frac{b}{M})$ and $I_\alpha(x_0) = [x_0 - \alpha, x_0 + \alpha]$. We see easily that the sequence $\{y_k(.)\}_{k \geqslant 0}$ constructed in (1.96) is uniformly bounded if the variable x is restricted to $x \in I_\alpha(x_0) \subseteq I_a(x_0)$. More precisely

$$y_k(x, \lambda) \in B(y_0, b), \forall (x, \lambda) \in I_\alpha(x_0) \times \Sigma, k \geqslant 0 \tag{1.98}$$

It will be proved by induction and assuming that (1.97) is satisfied for $k \geqslant 0$ we compute

$$\mid y_{k+1}(x, \lambda) - y_0 \mid \leqslant \int_{x_0}^{x_0 + |x - x_0|} f(t, \lambda, y_k(t, \lambda))dt \leqslant M\alpha \leqslant b \tag{1.99}$$

for any $\{(x, \lambda) \in I_\alpha(x_0) \times \Sigma\}$. Next step is to notice that $\{y_k(x, \lambda) : (x, \lambda) \in I_\alpha(x_0) \times \Sigma\}_{k \geqslant 0}$ is a Cauchy sequence in a Banach space $C(I_\alpha(x_0)) \times \sigma_j \mathbb{R}^n$ and it is implied by the following estimates

$$\mid y_{k+1}(x, \lambda) - y_k(x, \lambda) \mid \leqslant bL^k \frac{\mid x - x_0 \mid^k}{k!}, \forall (x, \lambda) \in I_\alpha(x_0) \times \Sigma, k \geqslant 0 \tag{1.100}$$

where $L = L(W) > 0$ is a Lipschitz constant associated with f and $W = I_\alpha(x_0) \times \Sigma \times B(y_0, b)$, a compact set of $I \times \Lambda \times G$. A verification of (1.99) uses the standard induction argument and for $k = 0$ they are proved in (1.98). Assuming (1.99) for k we compute $\mid y_{k+2}(x, \lambda) - y_{k+1}(x, \lambda) \mid \leqslant \int_{x_0}^{x_0 + |x - x_0|} \mid f(t, \lambda, y_{k+1}(t, \lambda)) - f(t, \lambda, y_k(t, \lambda)) \mid dt \leqslant L \int_{x_0}^{x_0 + |x - x_0|} \mid y_{k+1}(t, \lambda) - y_k(t, \lambda) \mid dt \leqslant L^{k+1} \frac{|x - x_0|^{k+1}}{k+1!}$ and (1.99) is verified.Rewrite (1.99)

$y_{k+1}(x, \lambda) = y_0 + (y_1(x, \lambda) - y_0) + ..., + y_{k+1}(x, \lambda) - y_k(x, \lambda) =$
$= y_0 + \Sigma_{j=0}^{k+1} u_j(x, \lambda)$, where $u_j = y_{j+1} - y_j$
and consider the following series of continuous functions

$$S(x, \lambda) = y_0 + \Sigma_{j=0}^{\infty} u_j(x, \lambda) \tag{1.101}$$

The series (1.100) is convergent in the Banach space $C(I_\alpha(x_0) \times \Sigma; \mathbb{R}^n)$ if it is bounded by a numerical convergent series and notice that each $\{u_j(x, \lambda) : (x, \lambda) \in I_\alpha(x_0) \times \Sigma\}$ of (1.100) satisfies (see(1.99))

$$\mid u_j(x, \lambda) \mid \leqslant b \frac{L^j \mid x - x_0 \mid^j}{j!} \leqslant b \frac{L.\alpha^j}{j!}, \ j \geqslant 0 \tag{1.102}$$

In conclusion, the series $S(x, \lambda)$ given in 1.101 is bounded by the following series

$$u = \mid y_0 \mid + b(1 + \frac{L\alpha}{1} + ... + \frac{L\alpha^k}{k!} + ...) = \mid y_0 \mid + b \exp L\alpha$$

and the sequence of continuous functions

$$\{y_k(x,\lambda) : (x,\lambda) \in I_\alpha(x_0) \times \Sigma\}_{k \geqslant 0}$$

constructed in (1.96) is uniformly convergent to a continuous function

$$y(x,\lambda) = \lim_{k\to\infty} y_k(x,\lambda) \text{ uniformly on } (x,\lambda) \in I_\alpha(x_0) \times \Sigma \tag{1.103}$$

It allows to pass $k \to \infty$ into integral equation (1.96) and we get

$$y(x,\lambda) = y_0 + \int_{x_0}^{x} f(t,\lambda,y(t,\lambda))dt, \forall (x,\lambda) \in I_\alpha(x_0) \times \Sigma \tag{1.104}$$

which proves the existence of the $C.P(f; x_0, y_0)$ solution.
Uniqueness. Let $y_1(x,\lambda) : J_1 \times \Sigma_1 \to G$ another solution of (1.94) satisfying $y_1(x_0) = y_0$, where J_1(compact set)$\subseteq I$ and Σ_1(*compact*) $\subseteq \Lambda$. Define a compact set $K_1 \subseteq G$ such that it contains all values of the continuous function

$$\{y_1(x,\lambda) : (x,\lambda) \in J_1 \times \Sigma_1\} \subseteq K_1$$

Denote

$$\widetilde{K} = B(y_0, b) \cap K_1, \widetilde{J} = I_\alpha(x_0) \cap J_1, \widehat{\Sigma} = \Sigma \cap \Sigma_1$$

and let

$$\widetilde{L} = L(\widetilde{J} \times \widetilde{\Sigma} \times \widetilde{K}) > 0$$

be the corresponding Lipschitz constant associated with f and compact set $\widetilde{W} = \widetilde{J} \times \widetilde{\Sigma} \times \widetilde{K}$.We have $\widetilde{J} = [x_0 - \delta_1, x_0 + \delta_2]$ for some $\delta_i \geqslant 0$ and

$$| \, y(x,\lambda) - y_1(x,\lambda) \, | \leqslant \widetilde{L} \int_{x_0}^{x_0 + |x-x_0|} | \, y(t,\lambda) - y_1(t,\lambda) \, | \, dt, \forall (x,\lambda) \in \widetilde{J} \times \widetilde{\Sigma} \tag{1.105}$$

For $\lambda \in \widetilde{\Sigma}$ fixed,denote $\varphi(x) = | \, y(x,\lambda) - y_1(x,\lambda) \, |$ and inequality (1.105) becomes

$$\varphi \leqslant \widetilde{L} \int_{x_0}^{t} \varphi(s)ds, t \in [x_0, x_0 + \delta_2] \tag{1.106}$$

which shows that $\varphi(t) = 0, \forall t \in [x_0, x_0 + \delta_2]$(see Gronwall Lemma in (1.1.1))
Similarly we get $\varphi(t) = 0, t \in [x_0 - \delta_1, x_0]$
and $\varphi(t) = 0 \; \forall \, t \in \widetilde{J}$, lead us to the conclusion $y(x,\lambda) = y_1(x,\lambda), \forall x \in \widetilde{J}$ and for an arbitrary fixed $\lambda \in \widetilde{\Sigma}$. The proof is complete. □

Remark 1.4.5. *The local Lipschitz continuity of the function f is essential for getting uniqueness of a C.P solution. Assuming that f is only a continuous function of $y \in G$,we can construct examples supporting the idea that a C.P solution is not unique. In this respect, consider the scalar function $f(y) = 2\sqrt{| \, y \, |}, y \in \mathbb{R}$ and the equation $\frac{dy(t)}{dt} = 2\sqrt{| \, y(t) \, |}$ with $y(0) = 0$. There are two C.P solution.$y_1(t) = 0, t \geqslant 0$ and $y_2(t) = t^2, t \geqslant 0$ where a continuous but not a Lipschitz continuous function was used.*

Comment. There is a general fixed point theorem which can be used for proving the existence and uniqueness of a Cauchy problem solution .In this respect we shall recall so called fixed point theorem associated with contractive mappings. Let $T : X \to X$be a continuous mapping satisfying

$$\rho(Tx, Ty) \leqslant \alpha\rho(x, y) \text{ for any } x, y \in X$$

where $0 < \alpha < 1$ is a constant and (X, ρ) is a complete metric space. A fixed point for the mapping T satisfies $T\widehat{x} = \widehat{x}$ and it can be obtained as a limit point of the following Cauchy sequence $\{x_n\}_{n\geqslant 0}$ defined by $x_{n+1} = Tx_n, n \geqslant 0$. By definition we get $\rho(x_{k+1}, x_k) = \rho(Tx_k, Tx_{k-1}) \leqslant \alpha\rho(x_k, x_{k-1}) \leqslant \alpha^k\rho(x_1, x_0)$ for any $k \geqslant 1$ and $\rho(Tx_{k+m}, Tx_k) \leqslant \Sigma_{j=1}^{m}\rho(x_{k+j}, x_{k+j-1}) \leqslant (\Sigma_{j=1}^{m}\alpha^{k+j-1})\rho(x_1, x_0)$, where $\Sigma_{k=0}^{\infty}\alpha^k = \frac{1}{1-\alpha}$ and $\{x_n\}_{n\geqslant 1}$ is Cauchy sequence.

1.4.2 Differentiability of Solutions with Respect to Parameters

In Theorem 1.4.4we have obtained the continuity property of the $C.P(f, x_0, y_0)$ solution with respect to parameters $\lambda \in \Lambda$ satisfying a differential system. Assume that the continuous function $f(x, \lambda, y) : I \times \Lambda \times G \to \mathbb{R}^n$ is continuously differentiable with respect $y \in G$ and $\lambda \in \Lambda$ i.e there exist continuous partial derivatives

$$\frac{\partial f(x, \lambda, y)}{\partial y}, \frac{\partial f(x, \lambda, y)}{\partial \lambda_j} : I \times \Lambda \times G \to \mathbb{R}^n, i \in \{1, ..., n\}, j \in \{1, ..., m\} \tag{1.107}$$

where $\Lambda \subseteq \mathbb{R}^m, G \subseteq \mathbb{R}^n$ are open sets.

Remark 1.4.6. *The assumption* (1.107) *leads us directly to the local Lipschitz property of f with respect to $y \in G$. In addition, let $y(x, \lambda) : I_\alpha(x_0) \times \Sigma \to B(y_0, b)$ be the $C.P(f; x_0, y_0)$ solution and define*
$F(x, z(x)) = f(x, \lambda, y(x, \lambda)), z(x) = (\lambda, y(x, \lambda))$ for each $x \in I_\alpha(x_0)$. Using (1.107) *we get that $F(x, z(x))$ satisfies the following differentiability property*

$$\begin{aligned} F(x, z''(x) - F(x, z'(x))) &= \frac{\partial f(x, \lambda', y(x, \lambda'))}{\partial y}[y(x, \lambda'') - y(x, \lambda')] \\ &+ \sum_{j=1}^{m} \frac{\partial f(x, \lambda', y(x, \lambda'))(\lambda''_j - \lambda'_j)}{\partial \lambda_j} + \theta(x, \lambda', \lambda'')(\mid y(x, \lambda'') \\ &- y(x, \lambda') \mid + \mid \lambda'' - \lambda' \mid) \end{aligned} \tag{1.108}$$

where

$$\lim_{\lambda'' \to \lambda'} \theta(x, \lambda', \lambda'') = 0$$

uniformly with respect to $x \in I_\alpha(x_0)$. To get (1.108) *we rewrite $F(x, z''(x)) -$*

$F(x,z'(x))$ as follows

$$F(x,z''(x)) - F(x,z'(x)) = \int_0^1 [\frac{dh}{d\theta}(x,\theta)]d\theta$$

where

$$h(x,\theta) = F(x,z'(x) + \theta(z''(x) - z'(x)), \theta \in [0,1], x \in I_\alpha(x_0)$$

The computation of the derivatives allows to see easily that (1.108) is valid. In addition using the assumption (1.107)we obtain the Lipschitz continuity of the solution $\{y(t,\lambda) : \lambda \in \sum\}$ and

$$| y(x,\lambda'') - y(x,\lambda') | \leqslant C | \lambda'' - \lambda' |, \forall x \in I_\alpha(x_0), \lambda', \lambda'' \in B(\lambda_0,\beta) = \Sigma \qquad (1.109)$$

where $C > 0$ is a constant. The property (1.109) is obtained applying lemma Gronwall for the integral inequality associated with the equation

$$y(x,\lambda'') - y(x,\lambda') = \int_0^x [F(t,z''(t)) - F(t,z'(t))]dt, x \in I_\alpha(x_0)$$

where $F(x,z(x))$ fulfils (1.108).

Theorem 1.4.7. *(differentiability of a solution)*
Let $f(x,\lambda,y) : I \times \Lambda \times G \to \mathbb{R}^n$ be given such that (1.107)is satisfied. Let$x_0 \in I, y_0 \in G, \lambda_0 \in \Lambda$ be fixed and define $a,b,\beta > 0$ such that $I_a(x_0) \subseteq I, B(y_0,b) \subseteq G, \sum = B(\lambda_0,\beta) \subseteq \Lambda$. Then there exist $\alpha > 0$ and $y(x,\lambda) : I_\alpha(x_0) \times \sum \to B(y_0,b)$ as a unique $C.P(f,x_0,y_0)$ solution for (1.94) such that for each $\widehat{\lambda} \in int\Sigma$ there exist $\frac{\partial y(x,\widehat{\lambda})}{\partial \lambda_j} = \widehat{y}_j(x), x \in I_\alpha(x_0), j \in 1,...,m$,satisfying the following linear system

$$\frac{dz}{dx} = \frac{\partial f(x,\widehat{\lambda},y(x,\widehat{\lambda}))}{\partial y} z + \frac{\partial f(x,\widehat{\lambda},y(x,\widehat{\lambda}))}{\partial \lambda_j},\ z(x_0) = 0 \qquad (1.110)$$

$x \in I_\alpha(x_0) = [x_0 - \alpha, x_0 + \alpha]$ for each $j \in \{1,...,m\}$.

Proof. By hypothesis, the conditions of Theorem 1.4.4 are fulfilled and let $y(x,\lambda) : I_\alpha(x_0) \times \Sigma \to B(y_0,b)$ be the unique solution of (1.94) satisfying $y(x_0,\lambda) = y_0, \lambda \in \Sigma$. Consider $\widehat{\lambda} \in int\Sigma$ and notice that differentiability of the solution with respect to parameters at $\lambda = \widehat{\lambda}$ is equivalent to showing

$$\lim_{\tau \to 0} E_\tau^j(x) = 0, \forall x \in I_\alpha(x_0), j \in 1,...,m \qquad (1.111)$$

where$E_\tau^j(x)$ is defined by ($e_1,...,e_m$ is canonical basis for $\mathbb{R}^m$)

$$E_\tau^j(x) = \frac{1}{\tau}[y(x,\widehat{\lambda} + \tau e_j) - y(x,\widehat{\lambda}) - \tau\widehat{y_j}(x)], \tau \neq 0 \qquad (1.112)$$

and $\widehat{y_j}(x), x \in I_\alpha(x_0)$, is the unique solution of (1.110). Using remark 1.4.6 for

$\lambda'' = \widehat{\lambda} + \tau e_g, \lambda' = \widehat{\lambda}$ we get

$$\begin{aligned} E_\tau^j(x) &= \int_{x_0}^{x} \frac{\partial f}{\partial y}(t, \widehat{\lambda}, y(t, \widehat{\lambda})) E_\tau^j(x) dt + \\ &+ \int_{x_0}^{x} (\theta(t, \widehat{\lambda} + \tau e_j \widehat{\lambda}) \frac{1}{\tau} \mid y(t, \widehat{\lambda} + \tau e_j - y(t, \widehat{\lambda}) \mid +1\} dt \end{aligned} \tag{1.113}$$

Denote $M_1 = C + 1$, where$\frac{1}{\tau} \mid y(t, \widehat{\lambda} + \tau e_j) - y(t, \widehat{\lambda})) \mid \leqslant C$ and let $L > 0$ be such that $\mid \frac{\partial f(t, \widehat{\lambda}, y(t, \widehat{\lambda}))}{\partial y} \mid \leqslant L \, \forall t \in I_\alpha(x_0)$. Then the following integral inequality is valid

$$\mid E_\tau^j(x) \mid \leqslant L \int_{x_0}^{x_0 + |x - x_0|} E_\tau^j(x) \mid dt + M_1 \int_{x_0}^{x_0 + \alpha} \mid \theta(t, \widehat{\lambda} + \tau e_j, \widehat{\lambda}) \mid dt, x \in I_\alpha(x_0) \tag{1.114}$$

Applying Lemma Gronwall, from (1.114) we obtain

$$\mid E_r^j(x) \mid \leqslant M_1 (\int_{x_0}^{x_0 + \alpha} \mid \theta(t, \widehat{\lambda} + \tau e_j, \widehat{\lambda}) \mid dt) \exp L\alpha \tag{1.115}$$

Using $\underset{\tau \to 0}{lim}\, \theta(t, \widehat{\lambda} + \tau e_j, \widehat{\lambda}) = 0$ uniformly of $t \in I_\alpha(x_0)$ (see(1.108)) and passing $\tau \to 0$ into (1.115) we obtain

$$\underset{r \to 0}{lim}\, E_\tau^j(x) = 0, \forall x \in I_\alpha(x_0), j \in 1, ..., m \tag{1.116}$$

The proof is complete. □

1.4.3 The Local Flow(Differentiability Properties)

Consider a continuous function $g(x, z) : I \times G \to \mathbb{R}^n$, where $I \subseteq \mathbb{R}$ and $G \subseteq \mathbb{R}^n$ are open sets. Define a new nonlinear system of differential equations

$$\frac{dz}{dx} = g(x, z), z(x_0) = \lambda \in B(z_0, \rho) \subseteq G \text{, where } x_0 \in I, z_0 \in G \tag{1.117}$$

are fixed and $\lambda \in B(z_0, \rho)$ is a variable Cauchy condition. Assume

$$\text{there exist continuous partial derivatives } \frac{\partial g(x, z)}{\partial z_i} : I \times G \to \mathbb{R}^n \; i \in 1, ..., n \tag{1.118}$$

The unique solution of (1.117), $z(x, \lambda) : I_\alpha(x_0) \times B(z_0, \rho) \to G$ satisfying $z(x_0, \lambda) = \lambda \in B(z_0, \rho) \subseteq \mathbb{R}^n$ will be called the local flow associated with the vector field g satisfying (1.118).

Theorem 1.4.8. *(differentiability of local flow) Consider that the vector field $g \in C(I \times G; \mathbb{R}^n)$ satisfies the assumption* (1.118)*. Then there exist $\alpha > 0$ and a continuously differentiable local flow $z(x, \lambda) : I_\alpha(x_0) \times B(z_0, \rho) \to G$ of g fulfilling the*

following properties

$$\text{for each}\,\widehat{\lambda} \in int\,B(z_0, \rho)\ \text{ there exists a nonsingular}\,(n \times n)\text{matrix} \tag{1.119}$$

$\widehat{Z}(x) = \frac{\partial z(x,\widehat{\lambda})}{\partial \lambda}, x \in I_\alpha(x_0)$, *satisfying*

$$\begin{cases} \frac{d\widehat{Z}(x)}{dx} = & \frac{\partial g(x,z(x,\widehat{\lambda}))}{\partial z}\widehat{Z}(x), \forall x \in I_\alpha(x_0) \\ \widehat{Z}(x_0) = & I_n \end{cases}$$

Proof. For $z_0 \in G$ fixed define $\Lambda = \{z \in G :\mid z - z_0 \mid < \rho + \epsilon\} = \text{int}B(z_0, \rho + \epsilon)$, where $\epsilon > 0$ is sufficiently small such that $\Lambda \subseteq G$. Associate a new vector field depending on parameter $\lambda \in \Lambda$

$$f(x, \lambda, y) = g(x, y + \lambda), x \in I, \lambda \in \Lambda, y \in \widehat{G} = G - B(z_0, \rho + \epsilon) \tag{1.120}$$

Let $b > 0$ be such that $B(0, b) \subseteq \widehat{G}$ and $B(z_0, \rho + \epsilon + b) \subseteq G$. Notice that according to (1.118) we get a smooth vector field f with respect to $(\lambda, y) \in \Lambda \times \widehat{G}$ and the following system of differential equation with parameters $\lambda = (\lambda_1, ..., \lambda_n) \subseteq \Lambda$

$$\frac{dy}{dx} = f(x, \lambda, y), y(x_0) = 0, y \in \widehat{G} \subseteq \mathbb{R}^n \tag{1.121}$$

satisfies the differentiability conditions of the Theorem 1.4.7. Let $y(x, \lambda) : I_\alpha(x_0) \times B(z_0, \rho) \to B(0, b) \subseteq \widehat{G}$ be the unique solution of (1.121) which is continuously differentiable on $(x, \lambda) \in I_\alpha(x_0) \times intB(z_0, \rho)$ and $y_{\widehat{j}}(x) = \frac{\partial y(x,\widehat{\lambda})}{\partial y}, x \in I_\alpha(x_0), \widehat{\lambda} \in intB(z_0, \rho)$,fulfils

$$\begin{cases} \frac{du}{dx} = & \frac{\partial f(x,\widehat{\lambda},y(x,\widehat{\lambda}))}{\partial y}u + \frac{\partial f(x,\widehat{\lambda},y(x,\widehat{\lambda}))}{\partial \widehat{\lambda}_j}, u \in \mathbb{R}^n, x \in I_\alpha(x_0) \\ u(x_0) = & 0 \end{cases} \tag{1.122}$$

for each $j \in 1, .., n$. Then $z(x, \lambda) = y(x, \lambda) + \lambda, x \in I_\alpha(x_0), \lambda \in B(z_0, \rho)$, is the unique continuously differentiable solution of the nonlinear system (1.117) with $z(x_0, \lambda) = \lambda$ and

$$\widehat{Z}(x) = \frac{\partial z(x, \widehat{\lambda})}{\partial \lambda} = \frac{\partial y(x, \widehat{\lambda})}{\partial \lambda} + I_n, x \in I_\alpha(x_0), \widehat{\lambda} \in intB(z_0, \rho) \tag{1.123}$$

where $\widehat{Y}(x) = \frac{\partial y(x,\widehat{\lambda})}{\partial \lambda}$ verifies the following system

$$\begin{cases} \frac{d\widehat{Y}(x)}{dx} = & \frac{\partial f(x,\widehat{\lambda},y(x,\widehat{\lambda}))}{\partial y}\widehat{Y}(x) + \frac{\partial f(x,\widehat{\lambda},y(x,\widehat{\lambda}))}{\partial \lambda}, x \in I_\alpha(x_0) \\ \widehat{Y}(x_0) = & \text{null matrix} \end{cases} \tag{1.124}$$

Using (1.123) and (1.124) we see easily that $\widehat{Z}(x_0) = I_n$ and

$$\begin{aligned}\frac{d\widehat{Z}(x)}{dx} &= \frac{d[\widehat{Y}(x)]}{dx} = \frac{\partial g(x, z(x, \widehat{\lambda}))}{\partial z}[\widehat{Z}(x) - I_n] + \frac{\partial g(x, z(x, \widehat{\lambda}))}{\partial z} \\ &= \frac{\partial g(x, z(x, \widehat{\lambda}))}{\partial z}\widehat{Z}(x), x \in I_\alpha(x_0) \qquad (1.125)\end{aligned}$$

The matrix satisfying (1.125) is a nonsingular one(see Liouville theorem) and the proof is complete. □

Remark 1.4.9. *Consider the nonsingular matrix $\widehat{Z}(x), x \in I_\alpha(x_0)$ given in the above theorem and define $\widehat{H}(x) = [\widehat{Z}(x)]^{-1}, x \in I_\alpha(x_0)$. Then $\widehat{H}(x), x \in I_\alpha(x_0)$ satisfies the following linear matrix system*

$$\begin{cases} \frac{d\widehat{H}(x)}{dx} = & -\widehat{H}(x)\frac{\partial g(x,z(x,\widehat{\lambda}))}{\partial z}, x \in I_\alpha(x_0) \\ H(x_0) = & I_n \end{cases} \qquad (1.126)$$

It can be proved by computing the derivative

$$\frac{d[\widehat{H}(x)\widehat{Z}(x)]}{dx} = [\frac{d\widehat{H}(x)}{dx}]\widehat{Z}(x) + \widehat{H}(x)[\frac{dZ}{dx}] = 0, x \in I_\alpha(x_0)$$

Exercise(differentiability with respect to $x_0 \in I$)
Let $g(x, z) : I \times G \to \mathbb{R}^n$ be continuously differentiable mapping with respect to $z \in G$.Let $x_0 \in I, z_0 \in G$ and $\beta > 0$ be fixed such that $I_\beta(x_0) = [x_0 - \beta, x_0 + \beta] \subseteq I$. Then there exist $\alpha > 0$ and a continuously differentiable mapping $z(x; s) : I_\alpha(s) \times I_\beta(x_0) \to G$ satisfying

$$\begin{cases} \frac{dz(x;s)}{dx} = & g(x, z(x; s)), x \in I_\alpha(s) = [s - \alpha, s + \alpha] \\ z(s; s) = & z_0, \text{ for each } s \in I_\beta(x_0) \end{cases} \qquad (1.127)$$

and $\widehat{z}(x) = \frac{\partial z(x,\widehat{s})}{\partial s}, x \in I_\alpha(\widehat{s})$ (for $\widehat{s} \in \text{int}I_\beta(x_0)$) fulfills the following linear system

$$\frac{d\widehat{z}(x)}{dx} = \frac{\partial g(x, z(x, \widehat{s}))}{\partial s}\widehat{z}(x), x \in I_\alpha(\widehat{s}), \widehat{z}(\widehat{s}) = -g(\widehat{s}, z_0) \qquad (1.128)$$

Hint. A system with parameters is associated as in Theorem 1.4.8 and it lead us to a solution

$$z(x, s) : I_\alpha(s) \times I_\beta(x_0) \to B(z_0, b) \subseteq G$$

of the following integral equation

$$z(x, s) = z_0 + \int_s^x g(t, z(t, s))dt, x \in I_\alpha(s), s \in I_\beta(x_0)$$

Take $\widehat{s} \in intI_\beta(x_0)$ and using a similar computation given in Theorem 1.4.7 we get

$\lim\limits_{\tau\to 0} E_\tau(x) = 0, x \in I_\alpha(\widehat{s})$, where

$$E_\tau(x) = \frac{1}{\tau}[z(x, \widehat{s}+\tau) - z(x, \widehat{s}) - \tau\widehat{z}(x)], \tau \neq 0$$

1.4.4 Applications(Using Differentiability of a Flow)

(a) The local flow $z(x,\lambda) : I_\alpha(x_0) \times B(z_0,\rho) \to \mathbb{R}^n$ defined in Theorem 1.4.7 preserve the volume of any bounded domain $D \subseteq B(z_0,\rho)$ for $\widehat{\lambda} \in D$ provided $Tr\frac{\partial g(x,z(x,\widehat{\lambda}))}{\partial z} = \Sigma_{i=1}^n \frac{\partial g_i(x,z(x,\widehat{\lambda}))}{\partial z_i} = 0, x \in I_\alpha(x_0), \widehat{\lambda} \in D$. In this respect, denote $D(x) = \{y \in \mathbb{R}^n, y = z(x,\widehat{\lambda}), \widehat{\lambda} \in D\}, x \in I_\alpha(x_0)$ and notice that $D(x_0) = D$. Using multiple integrals we compute $volD(x) = \int ... \int dy_1...dy_n$ which reduces to

$$D(x) = \int ... \int \mid det\frac{\partial z(x,\widehat{\lambda})}{\partial \lambda} \mid d\lambda_1...d\lambda_n$$

Using Liouville theorem and $Tr\frac{\partial g(x,z(x,\widehat{\lambda}))}{\partial z} = 0, \forall x \in I_\alpha(x_0), \widehat{\lambda} \in D$, we obtain $det\frac{\partial z(x,\widehat{\lambda})}{\partial \lambda} = 1$ for any $x \in I_\alpha(x_0), \widehat{\lambda} \in D$, where

$$\frac{\partial z(x,\widehat{\lambda})}{\partial \lambda}, x \in I_\alpha(x_0)$$

satisfies the linear system (1.119). As a result,$det]\frac{\partial z(x,\widehat{\lambda})}{\partial \lambda}] = 1, \forall x \in I_\alpha(x_0)$ and $\widehat{\lambda} \in D$, which proves that $volD(x) = volD, x \in I_\alpha(x_0)$

(b) The linear system (1.119) given in Theorem 1.4.8 is called the linearized system associated with (1.117).

If $g(x,z) = g(z)$ such that $g(z)$ satisfies a linear growth condition

$$\mid g(z) \mid \leqslant C(1+ \mid z \mid), \forall z \in \mathbb{R}^n \tag{1.129}$$

where $C > 0$ is a constant ,then the unique solution $z(x,\lambda), \lambda \in B(z_0,\rho)$ verifying

$$\frac{dz}{dx}(x,\lambda) = g(z(x,\lambda)), z(0) = \lambda \tag{1.130}$$

can be extended to the entire half line $x \in [0,\infty)$.

In addition, if $g(z_0) = 0$(z_0 is a stationary point)then the asymptotic behaviour of $z(x,\lambda)$ for $x \to \infty$ and $\lambda \in B(z_0,\rho)$ can be obtained analyzing the corresponding linear constant coefficients system

$$\frac{dz}{dx} = \frac{\partial g(z_0)}{\partial z} z, z(0) = \lambda \tag{1.131}$$

It will be assuming that

$$A = \frac{\partial g(z_0)}{\partial z} \text{ is a Hurwitz matrix i.e} \tag{1.132}$$

any$\lambda \in \sigma(A)(P(\lambda) = det(A - \lambda I_n) = 0)$ satisfies $Re\lambda < 0$, which implies $\mid \exp Ax \mid \leqslant M[\exp - wx]$ for some $M > 0, w > 0$.
We say that the nonlinear system (1.130) is locally asymptotically stable around the stationary solution z_0(or z_0 is locally asymptotically stable)if there exist $\rho > 0$ such that

$$\lim_{x \to \infty} z(x, \lambda) = 0, \forall \lambda \in B(z_0, \rho) \tag{1.133}$$

There is a classical result connecting (1.132) and (1.133).

Theorem 1.4.10. *(Poincare-Lyapunov) Assume that the $n \times n$ matrix A is Hurwitz such that*

$$\mid expAx \mid \leqslant M(exp - wx), \forall x \in [0, \infty)$$

for some constant $M > 0, w > 0$. Let $f(y) : \Omega \subseteq \mathbb{R}^n \times \mathbb{R}^n$ be a local Lipschitz continuous function such that Ω is an open set and $\mid f(y) \mid \leqslant L \mid y \mid, y \in \Omega(0 \in \Omega)$, where $L > 0$ is a constant. If $LM - w < 0$,then $y = 0$ is asymptotically stable for the perturbed system

$$\frac{dy}{dx} = Ay + f(y), y \in \Omega, y(0) = \lambda \in B(0, \rho) \subseteq \Omega \tag{1.134}$$

Here the unique solution $\{y(x, \lambda) : x \geqslant 0\}$ of(40) fulfils $\lim_{x \to \infty} y(x, \lambda) = 0$ for each $\lambda \in B(0, \rho)$ provided $\rho = \frac{\delta}{2M}$ and $B(0, \rho) \subseteq \Omega$.

Proof. Let$\lambda \in \Omega$ be fixed and consider the unique solution $\{y(x, \lambda) : x \in [0, T]\}$ satisfying (1.134). Notice that if λ is sufficiently small then $y(x, \lambda) : x \in [0, T]$,can be extended to the entire half line $x \in [0, \infty)$. In this respect, using the constant variation formula for $x \in [0, T]$ and $b(x) = f(y(x, \lambda))$ we get $y(x, \lambda) = (\exp Ax)\lambda + \int_0^x (\exp(x - t)A) f(y(t, \lambda))dt$,for any $x \in [0, T]$. Using the above given representation we see easily the following estimation

$$\mid y(x, \lambda) \mid \leqslant \mid \exp Ax \mid\mid \lambda \mid + \int_0^x \mid \exp(x - t)A \mid\mid f(y(t, \lambda)) \mid dt \leqslant$$

$$\leqslant \mid \lambda \mid M(\exp - wx) + \int_0^x LM(\exp - w(x - t)) \mid y(t, \lambda) \mid dt \text{for any} x \in [0, T]$$

Multiplying by $(\exp wx) > 0$, we obtain

$$(\exp wx) \mid y(x, \lambda \mid) \leqslant \mid \lambda \mid M + \int_0^x LM(\exp wt) \mid y(t, \lambda \mid) dt, x \in [0, T]$$

and applying Gronwall Lemma for $\varphi(x) = (\exp wx) \mid y(x, \lambda) \mid$ we get $\varphi(x) \leqslant \mid \lambda \mid$

$M(\exp LMx)$ for any $x \in [0,T]$. Take $\delta > 0$ and $\rho = \frac{\delta}{2M}$ such that $B(0,\delta) \subseteq \Omega$. It implies $\mid y(x,\lambda) \mid \leqslant \frac{\delta}{2}$ for any$x \in [0,T]$ if $\mid \lambda \mid \leqslant \rho$. In addition, $\mid y(T,\lambda) \mid \leqslant \frac{\delta}{2}\exp(-\alpha T)$, where $\alpha = w - LM > 0$ and choosing $T > 0$ sufficiently large such that $M\exp - \alpha T \leqslant 1$ we may extend the solution $y(x,\lambda), x \in [0,T]$ for $x \in [0,2T]$ such that

$$\mid y(2T,\lambda) \mid = \mid y(2T, y(T,\lambda)) \mid \leqslant M \mid y(T,\lambda) \mid \exp - \alpha T \leqslant (\frac{\delta}{2})$$

and

$$\mid y(x, y(T,\lambda)) \mid \leqslant M \mid y(T,\lambda) \mid \exp(LM - w)x \leqslant \frac{\delta}{2}$$

for any $x \in [T,2T]$. It shows that $y(x,\lambda)$ can be extended to the entire half line $x \in [0,\infty)$ if $\mid \lambda \mid \leqslant \rho = \frac{\delta}{2M}$ and $\delta > 0$ satisfies $B(0,\delta) \subseteq \Omega$
In addition the inequality established for $x \in [0,T]$ is preserved for the extended solution and $\mid y(x,\lambda) \mid \leqslant \mid \lambda \mid M\exp(LM - w)x \leqslant \frac{\delta}{2}\exp(LM - w)x$, for any $x \in [0,\infty)$, if $\mid \lambda \mid \leqslant \rho = \frac{\delta}{2}$. Passing to the limit $x \to \infty$, from the last inequality we get $\lim\limits_{x \to \infty} \mid y(x,\lambda) \mid = 0$ uniformly with respect to $\lambda \mid \leqslant \rho$ and the proof is complete. □

The following is a direct consequence of the above theorem.

Remark 1.4.11. *Let A be $n \times n$ Hurwitz matrix and $f(y) : \Omega(open) \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is a local Lipschitz continuous function where $0 \in \Omega$. If $\mid f(y) \mid \leqslant \alpha(\mid y \mid), \forall y \in \Omega$,where $\alpha(\mid y \mid) : [0,\infty) \to [0,\infty)$ satisfies $\lim\limits_{\gamma \to 0} \frac{\alpha(\gamma)}{\gamma} = 0$. Then the unique solution of the system*

$$\frac{dy}{dx} = Ay + f(y), y(0) = \lambda \in B(0,\rho), x \geqslant 0$$

is a asymptotically stable($\lim\limits_{x \to \infty} \mid y(x,\lambda \mid = 0)$) if $\rho > 0$ is sufficiently small.

Proof. Let $M \geqslant 1$ and $w > 0$ such that $\mid \exp Ax \mid \leqslant M(\exp - wx)$. Consider $L > 0$ such that $LM - w < 0$ and $\eta > 0$ with the property $\alpha(r) \leqslant Lr$ for any $r \in [0,\eta]$. Define $\widetilde{\Omega} = y \in \Omega; \mid y \mid < \eta$. Notice that $g(y) = Ay + f(y), y \in \widetilde{\Omega}$, satisfies the assumption of Poincare-Lyapunov theorem which allows to get the conclusion. □

We are in position to mention those sufficient conditions which implies that the stationary solution $z_0 \in \mathbb{R}^n$ of the system

$$\frac{dz}{dx} = g(z), g(z_0) = 0 \tag{1.135}$$

is asymptotically stable.Let $g(z) : D(open) \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be given such that $g(z_0) = 0$ and
(i) $g(z) : D \to \mathbb{R}^n$ is continuously differentiable.
(ii) The matrix $\frac{\partial g(z_0)}{\partial z} = A$ is Hurwitz.

Under the hypothesis (i) and (ii) we rewrite

$$g(y) = Ay + f(y), \text{where} y = z - z_0 \text{and} f(y) = g(z_0 + y) - g(z_0)$$

Consider the linear system

$$\frac{dy}{dx} = Ay + f(y), y \in \{y \in \mathbb{R}^n : | y | < \mu\} = \Omega$$

where $\mu > 0$ is taken such that $z_0 + \Omega \subseteq D$. Using (i) we get that f is locally Lipschitz continuous on Ω and the assumptions of the above given remark are satisfied. We see easily that

$$f(y) = \int_0^1 [\frac{\partial g(z_0 + \theta y)}{\partial z} - \frac{\partial g(z_0)}{\partial z}] y d\theta, y \in \Omega$$

and

$$| f(y) | \leqslant \alpha(| y |) = (\int_0^1 h(\theta, | y |) d\theta) | y |, y \in \Omega$$

Here $h(\theta, | y |) = \max_{|w| \leqslant |y|} | \frac{\partial g(z_0+\theta w)}{\partial z} - \frac{g(z_0)}{\partial z} |$ satisfies $\underset{|y|\to 0}{lim} h(\theta, | y |) = 0$ uniformly on $\theta \in [0, 1]$ and $\underset{\gamma\to 0}{lim} \frac{\alpha(\gamma)}{\gamma} = 0$ therefore, the assumptions of the above given remark are satisfied when considering the nonlinear system (1.135) and the conclusion will be stated as

Proposition 1.4.12. *Let $g(z) : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function and $z_0 \in D$ fixed such that $g(z_0) = 0$. Assume that the condition (i) and (ii) are fulfilled. Then the stationary solution $z = z_0$ of the system* (1.135) *is asymptotically stable, i.e* $\underset{x\to\infty}{lim} | z(x, \lambda) - z_0 | = 0$ *for any* $| \lambda - z_0 | \leqslant \rho$, *if* $\rho > 0$ *is sufficiently small,where* $z(x, \lambda) : x \geqslant 0$ *is the unique solution of* (1.135) *with* $z(0, \lambda) = \lambda$

Problem. *Prove Poincare-Lyapunov theorem replacing Hurwitz property of the matrix A with $\sigma(A + A^*) = \{\lambda_1, ...\lambda_d\}$ where $\lambda_i < 0, i \in 1, ..., d$.*

Comment(on global existence of a solution)

Theorem 1.4.13. *(global existence)Let $g(x, z) : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function which is locally Lipschitz continuous with respect to $z \in \mathbb{R}^n$. Assume that for each $T > 0$ there exists $C_T > 0$ such that*

$$| g(x, z) | \leqslant C_T(1+ | z |), \forall x \in [0, T],\ z \in \mathbb{R}^n$$

Then the unique $C.P(g; 0, z_0)$ solution $z(x, z_0)$ satisfying $\frac{dz(x;z_0)}{dx} = g(x, z(x, z_0))$, $z(0, z_0) = z_0$, *is defined for any* $x \in [0, \infty)$.

Proof. **(sketch)**It is used the associated integral equation

$$z(x, z_0) = z_0 + \int_0^x g(t, z(t, z_0)) dt, x \in [0, KT]$$

and the corresponding Cauchy sequence allows to get a unique solution $z_k(x, z_0) : x \in [0, KT]$ for each $K \geqslant 1$. The global solution will be defined as an inductive limit $\widetilde{Z}(z, z_0) : [0, \infty) \to \mathbb{R}^n, \widetilde{Z}(z, z_0) = Z_k(x, z_0)$, if $x \in [0, KT]$, for each $k \geqslant 0$. □

1.5 Gradient Systems of Vector Fields and their Solutions; Frobenius Theorem

Definition 1.5.1. *Let $X_j(p; y) \in \mathbb{R}^n$ for $p = (t_1, ..., t_m) \in D_m = \prod_1^m(-a_i, a_i), y \in V \subseteq \mathbb{R}^n$ be continuously differentiable ($X_j \in \mathcal{C}^1(D_m \times V; \mathbb{R}^n)$) for $j \in \{1, ..., m\}$.
We say that $\{X_1(p; y), ..., X_m(p; y)\}$ defines a gradient system of vector fields(or fulfils the Frobenius integrability condition) if
$\frac{\partial X_j}{\partial t_i}(p; y) - \frac{\partial X_i}{\partial t_j}(p; y) = [X_i(p; .), X_j(p; .)](y) \forall i, j \in \{1, ..., m\}$, where $[Z_1, Z_2](y) = \frac{\partial Z_1(y)}{\partial y} Z_2(y) - \frac{\partial Z_2(y)}{\partial y} Z_1(y)$(Lie bracket).*

1.5.1 The Gradient System Associated with a Finite Set of Vector Fields

Theorem 1.5.2. *Let $Y_j \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R}^n), x_0 \in \mathbb{R}^n, j \in \{1, ..., m\}$ be fixed .Then there exist $D_m = \prod_1^m(-a_i, a_i), V(x_0) \subseteq \mathbb{R}^n$ and $X_j(p; y) \in \mathbb{R}^n$
$X_j \in \mathcal{C}^1(D_{j-1} \times V; \mathbb{R}^n), j \in \{1, ..., m\}, X_1(p_1; y) = Y_1(y)$,such that*

$$\frac{\partial y}{\partial t_1} = Y_1(y), \frac{\partial y}{\partial t_2} = X_2(t_1; y), ..., \frac{\partial y}{\partial t_m} = X_m(t_1, ..., t_{m-1}; y) \tag{1.136}$$

is a gradient system ($\frac{\partial X_j(p_j; y)}{\partial t_i} = [X_i(p_i; .), X_j(p_j, .)](y), i < j$) and

$$G(p; x) = G_1(t) \circ ... \circ G_m(t_m(x)), p = (t_1, ..., t_m) \in D_m, x \in V(x_0) \tag{1.137}$$

is the solution for (1.136) satisfying Cauchy condition $y(0) = x \in V(x_0)$. Here $G_j(t; x)$ is the local flow generated by the vector field $Y_j, j \in \{1, ..., m\}$.

Proof. We shall use the standard induction argument and for $m = 1$ we notice that the equations (1.136) and (1.137) express the existence and uniqueness of a local flow associated with a nonlinear system of differential equation (see Theorem 1.4.8). Assume that for $(m-1)$ given vector fields $Y_j \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R}^n)$ the conclusions (1.136) and (1.137) are satisfied,i.e there exist

$$\widehat{X_2}(\widehat{y}) = Y_2, \widehat{X_3}(t_2, y), ..., \widehat{X_m}(t_2, ..., t_{m-1}; y) \tag{1.138}$$

continuously differentiable with respect to
$\widehat{p} = (t_2, ..., t_m) \in \prod_2^m (-a_i, a_i)$ and $y \in \widehat{V}(x_0) \subseteq \mathbb{R}^n$ such that

$$\frac{\partial y}{\partial t_2} = Y_2(y), \frac{\partial y}{\partial t_3} = \widehat{X_3}(t_2; y), ..., \frac{\partial y}{\partial t_m} = \widehat{X_m}(t_2, ..., t_{m-1}; y) \tag{1.139}$$

satisfying Cauchy condition $y(0) = x \in \widehat{V}(x_0)$. Recalling that (1.139) is a gradient system we notice

$$\frac{\partial \widehat{X_j}(\widehat{p}; y)}{\partial t_i} = [\widehat{X_i}(\widehat{p_i}, .), \widehat{X_j}(\widehat{p_j; .})](y), 2 \leqslant i < j = 3, ..., m \tag{1.140}$$

where $\widehat{p_i} = (t_2, ..., t_{i-1})$ Let vector fields $Y_j \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R}^n), j \in \{1, ..., m\}$ and denote

$$y = G(p; x) = G_1(t_1) \circ \widehat{G}(\widehat{p}; x), p = (t_1, \widehat{p}) \in \prod_1^m (-a_i, a_i), x \in V(x_0) \subseteq \widehat{V}(x_0) \tag{1.141}$$

where $G_1(t)(x)$ is the local flow generated by Y_1 and $\widehat{G}(\widehat{p}, x)$ is defined in (1.138). By definition $\frac{\partial G(p;y)}{\partial t_1} = Y_1(G(p; x))$ and to prove that $y = G(p; x)$ defined in (1.141) fulfils a gradient system we write $G_1(-t_1; y) = \widehat{G}(\widehat{p}; x)$ for $y = G(p; x)$.A straight computation shows

$$\frac{\partial G_1(-t_1; y)}{\partial y} \frac{\partial y}{\partial t_j} = \frac{\partial \widehat{G}(\widehat{p}; y)}{\partial t_j}, \text{ for} j = 2, ..., m \tag{1.142}$$

which is equivalent with writing
$\frac{\partial y}{\partial t_j} = H_1(-t_1; y)\widehat{X_j}(\widehat{p_j}; G_1(-t_1; y)), j = 2, ..., m.$
Here the matrix $H_1(\sigma; y) = [\frac{\partial G_1(\sigma;y)}{\partial y}]^{-1}$ satisfies

$\frac{dH_1}{d\sigma} = -H_1 \frac{\partial Y_1}{\partial x}(G_1(\sigma; y)), H_1(0; y) = I_n$(see Theorem 1.4.8). Denote

$$X_2(t_1; y) = H_1(-t_1; y)Y_2(G_1(-t_1; y))$$

$$X_j(t_1, ...j_{j-1}; y) = H_1(-t_1; y)\widehat{X_j}(t_2, ..., t_{j-1}; G_1(-t_1; y)), j = 3, ..., m$$

for $y \in V(x_0) \subseteq \widehat{V}(x_0)$ and $p = (t_1, ..., t_m) \in \prod_1^m (-a_i, a_i)$. With these notations, the system (1.142) is written as follows

$$\frac{\partial y}{\partial t_1} = Y_1(y), \frac{\partial y}{\partial t_2} = X_2(t_1; y), ... \frac{\partial y}{\partial t_m} = X_m(t_1, ..., t_{m-1}; y) \tag{1.143}$$

and to prove that (1.143) stands for a gradient system we need to show that

$$\frac{\partial X_j}{\partial t_i}(p_j; y) = [X_i(p_i, .), X_j(p_j, .)](y), 1 \leqslant i < j = 2, ..., m \tag{1.144}$$

For $j = 2$ by direct computation we obtain

$$\frac{\partial X_2}{\partial t_1}(t_1; y) = H_1(-t_1; y)[Y_1, Y_2](G(-t_1; y)) \tag{1.145}$$

and assuming that we know (see the exercise which follows)

$$\begin{aligned} H_1(-t_1; y)[Y_1, Y_2](G_1(t_1, y)) &= [H_1(-t_1; .)Y_1(G_1(t_1; .)), H_1(-t_1; .)Y_2(G_1(t_1; .))](y) \\ &= [Y_1(.), X_2(t_1; .)](y) \end{aligned} \tag{1.146}$$

we get

$$\frac{\partial X_2}{\partial t_1}(t_1; y) = [Y_1(.), X_2(t_1; .)](y) \tag{1.147}$$

The equation (1.147) stands for (1.144) when $i = 1$and $j = 2, 3, ...m$. It remains to show (1.144) for $j = 3, ..., m$ and $2 \leqslant i < j$. Using (1.140) and
$\frac{\partial X_j}{\partial t_i}(p_j; y) = H_1(-t_1; y)\frac{\partial \widehat{X_j}}{\partial t_i}(\widehat{p_j}; G_1(-t_1; y))$,
we obtain

$$\frac{\partial X_j}{\partial t_i}(p_j; y) = H_1(-t_1; y)[\widehat{X_i}(\widehat{p_i}; .), \widehat{X_j}(\widehat{p_j}; .)](G_1(-t_1; y)) \tag{1.148}$$

if $2 \leqslant i < i < j = 3, ..., m$. The right side in (1.148) is similar to that in (1.144) and the same argument used above applied to (1.148) allow one to write

$$\frac{\partial X_j}{\partial t_i}(p_j; y) = [X_i(p_i; .), X_j(p_j; .)](y) \tag{1.149}$$

for any $2 \leqslant i < j = 3, ..., m$, and the proof is complete. $\square$

Exercise 1. Let $X, Y_1, Y_2 \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R}^n)$ and consider the local flow $G(\sigma; x), \sigma \in (-a, a), x \in V(x_0) \subseteq \mathbb{R}^n$,and the matrix $H(\sigma; x) = [\frac{\partial G}{\partial x}(\sigma; x)]^{-1}$ determined by the vector field X(see Theorem 1.4.8). Then

$$H(-t; x)[Y_1, Y_2](G(-t; x)) == [H(-t; .)Y_1(G(-t; .)), H(-t; .)Y_2(g(-t; .))](x)$$

for any $t \in (-a, a), x \in V(x_0) \subseteq \mathbb{R}^n$.
Solution. The Lie bracket in the right hand side of the conclusion is computed using $H(-t; y) = \frac{\partial G}{\partial x}(t; G(-t; y))$ and using the symmetry of the matrices $\frac{\partial G_i}{\partial x^2}(t; x), i \in \{1, ..., n\}$,
we get the conclusion, where $G = (G_1, ..., G_n)$.
Exercise 2. Under the same conditions as in Exercise 1, prove that

$$H(-t; y)Y_1(G(-t; y)) = Y_1(y), \text{ provided} X = Y_1$$

Solution. By definition $H(0; y) = I_n$ and

$$X_1(t; y) = H(-t; y)Y_1(G(-t; y))$$

satisfies $X_1(0; y) = Y_1(y)$. In addition, applying the standard derivation of $X_1(t; y)$

we get $\frac{d}{dt}X_1(t;y) = 0$ which shows $X_1(t;y) = Y_1(y), t \in (-a,a)$ and the verification is complete.

1.5.2 Frobenius Theorem

Let $X_j \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R}^n), j \in \{1, ..., m\}$ be given and consider the following system of differential equations

$$\frac{\partial y}{\partial t_j} = X_j(y), j = 1, ..., m, y(0) = x \in V \subseteq \mathbb{R}^n \tag{1.150}$$

Definition 1.5.3. *A solution for* (1.150) *means a function* $G(p;x) : D_m \times V \to \mathbb{R}^n$ *of class* $\mathcal{C}^2(G \in \mathcal{C}^2(D_m \times V; \mathbb{R}^n))$ *fulfilling* (1.136) *for any* $p = (t_1, ..., t_m) \in D_m = \prod_1^m(-a_i, a_i)$ *and* $x \in V(openset) \subseteq \mathbb{R}^n$. *The system* (1.150) *is completely integrable on* $\theta(openset) \subseteq \mathbb{R}^n$ *if for any* $x_0 \in \theta$ *there exists a neighborhood* $V(x_0) \subseteq \theta$ *and a unique solution* $G(p;x), (p;x) \in D_m \times V(x_0)$ *of* (1.150) *fulfilling* $G(0;x) = x \in V(x_0)$

Theorem 1.5.4. *(Frobenius theorem) Let* $X_j \in \mathcal{C}^2(\mathbb{R}^n, {}^n), j \in \{1, ..., m\}$ *be given. Then the system* (1.150) *is completely integrable on*

$$\theta \text{ (open set) } \subseteq \mathbb{R}^n iff [Y_i, Y_j](x) = 0$$

for any $i, j \in \{1, ..., m\}, x \in \theta$, *where* $[Y_i, Y_j] =$ *Lie bracket. In addition, any local solution* $y = G(p;x), (p,x) \in D_m \times V(x_0)$ *is given by* $G(p;x) = G_1(t_1) \circ ... \circ G_m(t_m)(x)$,*where* $G_j(\sigma)(x)$ *is the local flow generated by* $X_j, j \in \{1, ..., m\}$.

Proof. For given $\{Y_1, ..., Y_m\} \subseteq \mathcal{C}^2(\mathbb{R}^n; \mathbb{R}^n)$ and $x_0 \in \theta \subseteq \mathbb{R}^n$ fixed we associate the corresponding gradient system

$$\frac{\partial y}{\partial t_1} = Y_1(y), \frac{\partial y}{\partial t_2} = \widehat{X_2}(t_1; y), ..., \frac{\partial y}{\partial t_m} = \widehat{X_m}(t_2, ..., t_{m-1}; y) \tag{1.151}$$

and its solution

$$y = G(p;x) = G_1(t_1) \circ ... \circ G_m(t_m)(x), p \in D_m = \prod_1^m(-a_j, a_j) \tag{1.152}$$

satisfying $G(0;x_0) = x_0 \in V(x_0)$(see Theorem 1.5.2) and $G(p;x) \in \theta$. By definition $\frac{\partial y}{\partial t_j} = X_j(t_1, ...t_{j-1}; y), j \in \{1, ..., m\}$, if $y = G(p;x_0)$ and the orbit given in (1.152) is a solution of the system (1.150)iff

$$X_j(t_1, ...t_{j-1}; y) = Y_j(y), j \in \{2, ..., m\}, y \in V(x_0) \subseteq \theta \tag{1.153}$$

To prove (1.153) we notice that

$$X_2(t_1;y) = H_1(-t_1;y)Y_2(G_1(-t_1;y)), \text{ for any } y \in V(x_0) \tag{1.154}$$

where $H_1(-t_1;y) = [\frac{\partial G_1}{\partial y}(-t_1;y)]^{-1}$ satisfies linear equation

$$\begin{cases} \frac{dH_1}{d\sigma}(\sigma;y) = & -H_1(\sigma;y)\frac{\partial Y_1}{\partial y}(G_1(\sigma;y)), \sigma \in (-a_1,a_1) \\ H_1(0;y) = & I_n \end{cases} \tag{1.155}$$

(see Remark (1.4.9)). Using (1.154) and (1.155), by a direct computation we get

$$\begin{cases} X_2(0;y) = & Y_2(y), y \in V(x_0) \\ \frac{dX_2(t_1;y)}{dt_1} = & H_1(-t_1;y)[Y_1,Y_2](G_1(-t_1;y)) \end{cases}$$

for any $y \in V(x_0) \subseteq \theta, t_1 \in (-a_1,a_1)$. In particular

$$X_2(t_1;y) = Y_2 \text{ for any } y \in V(x_0) \text{iff} [Y_1,Y_2](y) = 0 \text{ for any } \ y \in V(x_0) \tag{1.156}$$

A similar argument can be used for proving that

$$\begin{aligned} X_j(t_1,...,t_{j-1};y) \quad = \quad & Y_j(y), \text{ for any } y \in V(x_0), j \geqslant 2\, iff [Y_i,Y_j](y) = 0 \\ & \text{for any } 1 \leqslant i \leqslant j-1, y \in V(x_0) \end{aligned} \tag{1.157}$$

and the conclusion of the system (1.150) implies

$$[Y_i,Y_j](y) = 0, \forall y \in V(x_0) \subseteq \theta, i,j \in \{1,...,m\} \tag{1.158}$$

The reverse implication, $\{Y_1,...Y_m\} \subseteq \mathcal{C}^2(\mathbb{R}^n,\mathbb{R}^n)$ are commuting on $\theta(open) \subseteq \mathbb{R}^n$ implies that the system is completely integrable on θ will be proved noticing t(1.150) hat

$$G_i(t_i) \circ G_j(t_j)(y) = G_j(t_j) \circ G_i(t_i)(y), \text{for any } i,j \in \{1,...,m\}, y \in V(x_0) \tag{1.159}$$

if (1.155)is assumed. Here $G_j(\sigma)(y)$ is the local flow generated by the vector field $Y_j, j \in \{1,...,m\}$. It remains to check that the orbit (1.152) is the unique solution of the system (1.150) and in this respect we notice that

$$\varphi(\theta;x) = G(\theta p;x), \theta \in [0,1] \tag{1.160}$$

satisfies the following system of ordinary differential equations

$$\begin{cases} \frac{d\varphi}{d\theta}(\theta;x) = & \Sigma_{j=1}^m t_j Y_j(\varphi(\theta;x)) = g(p,\varphi(\theta;x)) \\ \varphi(0;x) = & x, p = (t_1,...,t_m) \in D_m \end{cases} \tag{1.161}$$

Here $g(p,y)$ is a locally Lipschitz continuous function with respect to $y \in \theta$ and $\{\varphi(\theta;x) : \theta \in [0,1]\}$ is unique. The proof is complete. $\square$

Comment(total differential equation and gradient system)
For a continuously differentiable function $y(p) : D_m = \prod_1^m(-a_k, a_k) \to \mathbb{R}^n$ define the corresponding differential as follows

$$dy(p) = \Sigma_{j=1}^m \frac{\partial y(p)}{\partial t_j} dt_j$$

where

$$p = (t_1, ..., t_m) \in D_m$$

Let$\{Y_1, ..., Y_m\} \subseteq C_2(\mathbb{R}^n, \mathbb{R}^n)$ be given and consider the following equation (using differentials)
$dy(p) = \Sigma_{j=1}^m Y_j(y(p))dt_j, p \in D_m, y(0) = x \in V(x_0) \subseteq \mathbb{R}^n.\theta$ solution for the last equation implies that
$\frac{\partial y}{\partial y} = Y_j(y), j \in \{1, ..., m\}, p = (t_1, ..., t_m) \in D_m$ is a gradient system for $y \in V(x_0)$ and it can be solved using Frobenius theorem((1.5.4)). In addition, for an analytic function of $z \in D \subseteq \mathbb{C}$,
$w(z) : D \to \mathbb{C}^n$ defines the corresponding differential
$dw(z) = dw_1(x, y) + idw_2(x, y), z = a + iy \in D$
where $w(z) = w_1(x, y) + iw_2(x, y)$and
$v(x, y) = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}(x, y) : D_2 = \prod_1^2(-a_k, a_k) \to \mathbb{R}^{2n}$ is an analytic function.Let $f(w) : \Omega \subseteq \mathbb{C}^n \to \mathbb{C}^n$ be given analytic function and associate the following equation (using differentials)
$dw(z) = f(w(z))dz = [f_1(v(x, y))dx - f_2(v(x, y))dy] +$
$+ i[f_2(v(x, y))dx + f_2(v(x, y))dy]$ where
$f(w) = f_1(v) + if_2(v)$ and f_1, f_2 are real analytic functions of $v = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in D_{2n} = \prod_{k=1}^{2n}(-a_k, a_k)$. To solve the last complex equation we need to show that $Y_1(v) = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}(v)$ and $Y_2(v) = \begin{pmatrix} -f_1 \\ f_2 \end{pmatrix}(v)$are commuting on $v \in D_{2n}$ which shows the system $\frac{\partial v}{\partial x} = Y_1(v), \frac{\partial v}{\partial y} = Y_2(v)$ is completely integrable on D_2.

1.6 Appendix

(a_1)Assuming that $f(x, y) : I \times G \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function, Peano proved the following existence theorem

Theorem 1.6.1. *(Peano) Consider the following system of ODE*

$$\frac{dy}{dx} = f(x, y), y(x_0) = y_0 \text{ where } (x_0, y_0) \in I \times G$$

is fixed and $\{f(x, y) : (x, y) \in I \times G\}$ is a continuous function. Then there exist an interval $J \subseteq I$ and a continuously derivable function $y(x) : J \to G$ satisfying

$y(x_0) = y_0$ *and* $\frac{dy}{dx}(x) = f(x, y(x)), x \in J$.

Proof. It relies on Arzela-Ascoli theorem which is refering to so called compact sequence
$\{y_k(x), x \in D\}_{k\geqslant 1} \subseteq C(D; \mathbb{R}^n)$
satisfying

$$(i)\ \text{A boundedness condition} \quad | y_k(x) | \leqslant M, \forall x \in D, k \geqslant 0$$

$$(ii) \quad \text{they are equally uniformly continuous, i.e for any } \epsilon > 0 \text{ there exists } \delta(\epsilon) > 0$$
$$\text{such that } | x'' - x' | < \delta(\epsilon) \text{ implies } | y_k(x'') - y_k(x') | < \epsilon \, \forall \, x', x'' \in D, k \geqslant 1$$

If a sequence $\{y_k(x) : x \in D\}_{k\geqslant 1}$ satisfy (i) and (ii) then Arzela-Ascoli theorem allows to find a subsequence $\{y_{k_j}(x) : x \in D\}_{j\geqslant 1}$ which is uniformly convergent on the compact set D. Let $a > 0$ and $b > 0$ such that $I_a = [x_0 - a, x_0 + a] \subseteq I$ and $B(y_0, b) \subseteq G$. Define $M = \max\{| f(x, y) |: (x, y) \in I_a \times B(y_0, b)\}$ and let $\alpha = min(a, \frac{b}{M})$. Construct the following sequence of continuous functions

$$y_k(x) = [x_0, x_0 + \alpha] \to G, y_k(x) = y_0, \text{for} x \in [x_0, x_0 + \frac{\alpha}{k}]$$

(iii)

$$y_k(x) = y_0 + \int_{x_0 + \frac{\alpha}{k}}^{x} f(t, y_k(t - \frac{\alpha}{k})), \text{if } \ x \in [x_0 + \frac{\alpha}{k}, x_0 + \alpha] \text{for each } \ k \geqslant 1$$

$\{y_k(x) : x \in [x_0, x_0 + \alpha]\}$ is well defined because it is constructed on the interval $[x_0 + m\frac{\alpha}{k}, x_0 + (m + 1)\frac{\alpha}{k}]$ using its definition on the preceding interval $[x_0 + (m - 1)\frac{\alpha}{k}, x_0 + m\frac{\alpha}{k}]$. It is easily seen that$\{y_k(x) : x \in [x_0, \alpha]\}_{k\geqslant 1}$ is uniformly bounded and $| y_k(x) - y_0 | \leqslant M\alpha \leqslant b$.
In addition, $| y_k(x'') - y_k(x') | = 0$, if $x', x'' \in [x_0, x_0 + \frac{\alpha}{k}]$ and $| y_k(x'') - y_k(x') | \leqslant M | x'' - x' |$ if $x', x'' \in [x_0 + \frac{\alpha}{k}, x_0 + \alpha]$ which implies $| y_k(x'') - y_k(x') | \leqslant M | x'' - x' |$ for any $x', x'' \in [x_0, x_0 + \alpha], k \geqslant 1$.
Let $\widehat{y}(x) : [x_0, x_0 + \alpha] \to B(y_0, b) \subseteq G$ be the continuous function obtained using a subsequence$\{y_{k_j}(x) : x \in [x_0, x_0 + \alpha]\}_{j\geqslant 1}, \widehat{y}(x) = \underset{j\to\infty}{lim}\, y_{k_j}(x)$ uniformly with respect to $x \in [x_0, x_0 + \alpha]$. By passing $j \to \infty$ into the equation (iii) written for $k = k_j$ we get

$$\widehat{y}(x) = y_0 + \int_{x_0}^{x} f(t, \widehat{y(t)})dt, \forall x \in [x_0, x_0 + \alpha]$$

and the proof is complete. □

(a_2). Ordinary differential equations with delay
Let $\sigma > 0$ be fixed and consider the following system of ODE

$$\begin{cases} \frac{dy}{dx}(x) = & f(x, y(x), y(x - \sigma)), x \in [0, a] \\ y(\sigma) = & \varphi(\sigma), \sigma \in [-\sigma, \sigma] \end{cases} \tag{1.162}$$

where $\varphi \in C([-\sigma, 0]; \mathbb{R}^n)$ is fixed as a continuous function(Cauchy condition),and $f(x, y, z) : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function admitting first order continuous partial derivatives $\partial_{y_i} f(x, y, z) : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n, i \in \{1, ..., n\}$.
The above given assumptions allow us to construct a unique local solution satisfying (1.162) and it is done starting with the system (1.162) on the fixed interval $x \in [o, \sigma]$

$$\begin{cases} \frac{dy}{dx} = & f(x, y(x), \varphi(x - \sigma)), x \in [0, \sigma] \\ y(0) = & \varphi(0) \end{cases} \tag{1.163}$$

which satisfies the condition of Cauchy-Lipschitz theorem. In order to make sure that the solution of (1.163) exists for any $x \in [0, \sigma]$ we need to assume a linear growth of f with respect to the variable y uniformity of (x, z) in compact sets,i.e
(*)$| f(x, y, z) | \leqslant C(1+ | y |) \forall \in \mathbb{R}^n$ and $(x, z) \in J \times K(compact) \subseteq I \times \mathbb{R}^n$,where the constant $C > 0$ depends on the arbitrary fixed compact $J \times K$.
Any function f satisfying
(**)$f(x, y, z) = A(x, z)y + b(x, z), (x, z) \in I \times \mathbb{R}^n$ where the matrix $A(x, z) \in M_{n\times n}$ and the vector $b(x, z) \in \mathbb{R}^n$ are continuous functions will satisfy the necessary conditions to extend the solution of (1.163) on any interval $[m\sigma, (m + 1)\sigma], m \geqslant 0$

Exercises(Linear integrable system)
(1) Formulate Frobenius theorem ((1.5.4)). Rewrite the content of Frobenius theorem using linear vector fields $Y_i(y) = A_i y$, where $A_i \in M_{n\times n}(\mathbb{R}), i \in \{1, ..., n\}$.
(2) For any matrices $A, B \in M_{n\times n}(\mathbb{R})$ define the meaning of the exponential mapping $(\exp t ad_A)(B)$, where ad_A(adjoint mapping associated with A)is the linear mapping $ad_A : M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R})$, defined by $ad_A(B) = AB - BA = [A, B]$(Lie bracket of (A, B)).
(3) Show that $(\exp tA)B(\exp - tA) = (\exp t ad_A)(B)$, for any $A, B \in M_{n\times n}(\mathbb{R})$, $t \in \mathbb{R}$ where $(\exp t ad_A)$ is the linear mapping defined in (2).
(4)Rewrite the gradient system ((1.5.2))when the vector fields $Y_i(y) = A_i y, i \in \{1, ..., m\}$,are linear. Show that,in this case the vector fields with parameters are the following
$Y_1(y) = A_1 y, X_2(t_1; y) = (\exp t_1 ad_{A_1})(A_2)y, ..., X_m(t_1, ..., t_{m-1}; y) = [\exp t_1 ad_{A_1} \circ (\exp t_2 ad_{A_2}) ... \circ (\exp t_{m-1} ad_{A_{m-1}})(A_m)]y$,such that the corresponding gradient system
$\frac{\partial y}{\partial t_1} = Y_1(y), \frac{\partial y}{\partial t_2} = X_2(t_1, y), ..., \frac{\partial y}{\partial t_m} = X_m(t_1, ..., t_{m-1}; y)$ has the unique solution $y(t_1, ..., t_m; y_0) = (\exp t_1 A_1)...(\exp t_m A_m)y_0$.

Bibliographical Comments

The entire Chapter 1 is presented with minor changes as in the [12].

Chapter 2

First Order Partial Differential Equation

Let $D \subseteq \mathbb{R}^n$ be an open set and $f(y) : D \to \mathbb{R}^n$ is a continuous function.Consider the following nonlinear ODE

$$\frac{dy}{dt} = f(y) \tag{2.1}$$

Definition 2.0.2. *Let $D_0 \subseteq D$ be an open set. A function $u(y) : D_0 \to \mathbb{R}$ is called first integral for* (2.1) *on D_0 if*

1. *u is nonconstant on D_0,*
2. *u is continuously differentiable on $D_0(u \in \mathcal{C}^1(D_0; \mathbb{R}))$,*
3. *for each solution $y(t) : I \subseteq \mathbb{R} \to D_0$ of the system* (2.1) *there exists a constant $c \in \mathbb{R}$ such that $U(y(t)) = c, \forall\, t \in I$.*

Example 2.0.1. *A Hamilton system is described by the following system ofODE*

$$\frac{dp}{dt} = \frac{\partial H}{\partial q}(p, q), \frac{dq}{dt} = -\frac{\partial H}{\partial p}(p, q), p(0) = p_0, q(0) = q_0 \tag{2.2}$$

where $H(p, q) : G_1 \times G_2 \subseteq \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable and $G_i \subseteq \mathbb{R}^n, i \in \{1, 2\}$, are open sets. Let $\{(p(t), q(t)) : t \in [0, T]\}$ be solution of the Hamilton system (2.1) *and by a direct computation, we get*

$$\frac{d}{dt}[H(p(t), q(t))] = 0 \text{ for any } t \in [0, T]$$

It implies $H(p(t), q(t)) = H(p_0, q_0) \forall\, t \in [0, T]$ *and* $\{H(p, q) : (p, q) \in G_1 \times G_2\}$ *is a first integral for* (2.2)

Theorem 2.0.3. *Let* $f(y) : D \to \mathbb{R}^n$ *be a continuous function,* $D_0 \subseteq D$ *an open subset and consider a nonconstant continuously differentiable function* $U \in \mathcal{C}^1(D_0, \mathbb{R})$*. Then* $\{U(y) : y \in D_0\}$ *is first integral on* D_0 *of* ODE *iff the following differential equality*

$$< \frac{\partial u(y)}{\partial y}, f(y) >= \Sigma_{i=1}^{n} \frac{\partial u}{\partial y_i}(y) f_i(y) = 0, \forall y \in D_0 \tag{2.3}$$

is satisfied, where

$$f = (f_1, ..., f_n), \frac{\partial u}{\partial y} = (\frac{\partial u}{\partial y_1}, ..., \frac{\partial u}{\partial y_n})$$

Proof. Let $\{u(y) : y \in D_0\}$ be a continuously differentiable first integral ofODE(2.1) and consider that $y(t, y_0) : (-\alpha, \alpha) \to D_0$ is a solution of (2.1) verifying $y(0, y_0) = y_0 \in D_0$. By hypothesis, $u(y(t, y_0)) = c = u(y_0)$, for any $t \in (-\alpha, \alpha)$ and by derivation, we get

$$0 = \frac{d}{dt}[u(y(t, y_0))] =< \frac{\partial u}{\partial y}(y(t, y_0)), f(y(t, y_0)) >, \text{for any}\, t \in (-\alpha, \alpha) \tag{2.4}$$

In particular, for $t = 0$ we obtain the conclusion (2.3) for an arbitrary fixed $y_0 \in D_0$. The reverse implication uses the equality(2.3) and define $\varphi(t) = u(y(t)), t \in (-\alpha, \alpha)$ where $u \in \mathcal{C}^1(D_0, \mathbb{R})$ fulfils (2.3) and $\{y(t) : t \in (-\alpha, \alpha)\}$ is a solution of (2.1). It follows that

$$\frac{d\varphi}{dt}(t) =< \frac{\partial u}{\partial y}(y(t)), f(y(t)) >= 0, t \in (-\alpha, \alpha)$$

and

$$\varphi(t) = constant, t \in (-\alpha, \alpha)$$

The proof is complete. □

Let $\Omega \subseteq \mathbb{R}^{n+1}$ be an open set and consider that

$$f(x, u) : \Omega \to \mathbb{R}^n, L(x, u) : \Omega \to \mathbb{R} \tag{2.5}$$

are given continuously differentiable function, $f \in \mathcal{C}^1(\Omega; \mathbb{R}^n), L \in \mathcal{C}^1(\Omega; \mathbb{R})$.

Definition 2.0.4. *A causilinear first order* PDE *is defined by the following differential equality*

$$< \frac{\partial u(x)}{\partial x}, f(x, u(x)) >= L(x, u(x)), x \in D(open) \subseteq \mathbb{R}^n \tag{2.6}$$

where $f \in \mathcal{C}^1(\Omega; \mathbb{R}^n), L \in \mathcal{C}^1(\Omega; \mathbb{R})$ *are fixed and* $u(x) : D \to \mathbb{R}$ *is an unknown continuously differentiable function such that* $(x, u(x)) \in \Omega, \forall x \in D$

A solution for (2.6) means a function $u \in \mathcal{C}^1(D;\mathbb{R})$ such that $(x, u(x) \in \Omega)$ and (2.6)is satisfied for any $x \in D$

Remark 2.0.5. *Assuming that $L(x,u) = 0$ and $f(x,u) = f(x)$ for any $(x,u) \in D \times \mathbb{R} = \Omega$ then PDE defined in* (2.6) *stands for the differential equality*(2.3) *in Theorem 2.0.3 defining first integral for ODE* (2.1). *In this case, the differential system* (2.1)*is called Cauchy characteristic system associated with linear PDE*

$$< \frac{\partial u(x)}{\partial x}, f(x) >= 0, x \in D \subseteq \mathbb{R}^n \tag{2.7}$$

Proposition 2.0.6. *Let $D_0 \subseteq D$ be an open subset and consider a nonconstant continuously differentiable scalar function $u(x) : D_0 \to \mathbb{R}$. Then $\{u(x) : x \in D_0\}$ is a solution for the linear PDE* (2.7) *iff $\{u(x) : x \in D_0\}$ is a first integral for ODE* (2.1) *on D_0.*

Definition 2.0.7. *Let $a \in D \subseteq \mathbb{R}^n$ be fixed and consider m first integrals $\{u_1(y), ..., u_m(y) : y \in V(a) \subseteq D\}$ for ODE* (2.1) *which are continuously differentiable where $V(a)$ is a neighborhood of a. We say that $u_i \in \mathcal{C}^1(V(a);\mathbb{R}), i \in \{1, ..., m\}$ are independent first integrals of* (2.1) *if $rank(\frac{\partial u_1}{\partial y}(a), ..., \frac{\partial u_m}{\partial y}(a)) = m$*

Remark 2.0.8. *Let $f \in \mathcal{C}^1(D;\mathbb{R}^n)$ and $a \in D$ be fixed such that $f(a) \neq 0$. Then there exist a neighborhood $V(a) \subseteq D$ and $(n-1)$ independent first integrals $\{u_1(y), ..., u_{n-1}(y) : y \in V(a)\}$ of the system* (2.1).

Proof. Using a permutation of the coordinates $(f_1, ..., f_n) = f$ we say that $f_n(a) \neq 0$ (see $f(a) \neq 0$). Denote $\Lambda = \{(\lambda_1, ..., \lambda_{n-1}) \in \mathbb{R}^{n-1} : (\lambda_1, ..., \lambda_{n-1}, a_n) \in D\}$ and consider $\{F(t,\lambda) : (t,\lambda) \in [0,T] \times \Sigma\}$ as the local flow associated with ODE (2.1)

$$\begin{cases} \frac{dF}{dt}(t,\lambda) = & f(F(t,\lambda)), t \in [-\alpha,\alpha], \lambda \in \Sigma \subseteq \Lambda \\ F(0,\lambda) = & (\lambda_1, ..., \lambda_{n-1}, a_n) \end{cases} \tag{2.8}$$

where $\Sigma \subseteq \Lambda$ is a compact subset and $(a_1, ..., a_{n-1}) \in int\Sigma$. Using the differentiability properties of the local flow $\{F(t,\lambda) : (t,\lambda) \in [-\alpha,\alpha] \times \Sigma\}$, we get

$$det \begin{pmatrix} \frac{\partial F}{\partial t} & \frac{\partial F}{\partial \lambda_1} & . & . & . & \frac{\partial F}{\partial \lambda_{n-1}} \end{pmatrix} .(0, a_1, ..., a_{n-1}) =$$

$$= det \begin{pmatrix} f_1(a) & 1 & 0 & . & . & . & 0 \\ & 0 & 1 & & & & . \\ & . & 0 & & & & . \\ . & . & . & & & & . \\ . & . & . & & & & 0 \\ . & & . & & & & 1 \\ f_n(a) & 0 & 0 & . & . & . & 0 \end{pmatrix} = (-1)^{n+1} f_n(a) \neq 0$$

and the conditions for applying an implicit functions theorem are fulfilled when the algebraic equations

$$F(t,\lambda) = y, \ \ y \in V(a) \subseteq D, F(0, a_1, ..., a_{n-1}) = a \tag{2.9}$$

are considered. It implies that there exist $t = u_0(y), \lambda_i = u_i(y), i \in \{1, ..., n-1\}, y \in V(a)$, which are continuously differentiable such that

$$F(u_0(y), u_1(y), ..., u_{n-1}(y)) = y, \forall y \in V(a) \tag{2.10}$$

where $u_0(y) \in (-\alpha, \alpha)$ and $\lambda(y) = (u_1(y), ..., u_{n-1}(y)), \lambda \in \Sigma$, for any $y \in V(a)$.
In addition, $\{u_0(y), ..., u_{n-1}(y) : y \in V(a)\}$ is the unique solution satisfying (2.10) and

$$u_0(F(t, \lambda)) = t, u_i(F(t.\lambda)) = \lambda_i, i \in \{1, ..., n-1\} \tag{2.11}$$

Using (2.11) we get easily that $\{u_1(y), ..., u_{n-1}(y) : y \in V(a)\}$ are $(n-1)$ first integrals for ODE (2.1) which are independent noticing that

$$< \frac{\partial u_i}{\partial y}(a), \frac{\partial F}{\partial \lambda_j}(0, \lambda) >= \delta_{ij}$$

where

$$\frac{\partial F}{\partial \lambda_j}(0, \lambda) = e_j \in \mathbb{R}^n, \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq 0, \, j \in \{1, ..., n-1\} \end{cases}$$

and $\{e_1, ..., e_n\} \subseteq \mathbb{R}^n$ is the canonical basis. In conclusion,

$$< \frac{\partial u_i}{\partial y}(a), e_j >= \delta_{ij}$$

for any

$$i, j \in \{1, ..., n-1\}$$

and

$$rank \begin{pmatrix} \frac{\partial u_1}{\partial y}(a) & . & . & . & \frac{\partial u_{n-1}}{\partial y}(a) \end{pmatrix} = n - 1$$

The proof is complete. □

Theorem 2.0.9. *Let $f(y) : D \to \mathbb{R}^n$ be a continuously differentiable function such that $f(a) \neq 0$ for fixed $a \in D$. Let $\{u_1(y), ..., u_{n-1}(y) : y \in V(a) \subseteq D\}$ be the $(n-1)$ first integrals constructed in Theorem 2.0.8. Then for any first integral of* (2.1) *$\{u(y) : y \in V(a)\}, u \in \mathcal{C}^1(V, \mathbb{R})$, there exists an open set $\theta \subseteq \mathbb{R}^{n-1}$ and $h \in \mathcal{C}^1(\theta, \mathbb{R})$ such that $u(y) = h(u_1(y), ..., u_{n-1}(y)), y \in V_1(a) \subseteq V(a)$.*

Proof. By hypothesis $\{u(y) : y \in V(a)\}$ is a continuously differentiable first integral of ODE (2.1) and we get

$$u(F(t, \lambda_1, ..., \lambda_{n-1})) = h(\lambda_1, ..., \lambda_{n-1}), \forall t \in [-\alpha, \alpha] \tag{2.12}$$

provided that $\{F(t, \lambda) : t \in [-\alpha, \alpha], \lambda \in \Sigma\}$ is the local flow associated with ODE (2.1) satisfying $F(0, \lambda_1, ..., \lambda_{n-1}) = h(\lambda_1, ..., \lambda_{n-1}, a_n)$(see (2.8) of Theorem 2.0.8). As far as $F \in \mathcal{C}^1((-\alpha, \alpha) \times \Sigma; \mathbb{R}^n)$ and $u \in \mathcal{C}^1(V(a), \mathbb{R})$ we get $h \in \mathcal{C}^1(\theta, \mathbb{R})$ fulfilling (2.12) where $(a_1, ..., a_{n-1}) \in \theta(openset) \subseteq \Sigma$. Using (2.10) in Theorem 2.0.8 we obtain

$$F(u_0(y), ..., u_{n-1}(y)) = y \in V(a)$$

and rewrite (2.12) for

$$t = u_0(y), \lambda_i = u_i(y).i \in \{1, ..., n-1\}$$

we get

$$u(y) = h(u_1(y), ..., u_{n-1}(y)) \, \text{for some} \, y \in V_1(a) \subseteq V(a) \tag{2.13}$$

where $(u_1(y), ..., u_{n-1}(y)) \in \theta$, $y \in V_1$. The proof is complete. $\square$

Remark 2.0.10. *The above given consideration can be extended to non autonomous ODE*

$$\frac{dy}{dt} = f(t, y), t \in\in I \subseteq \mathbb{R}, y \in G \subseteq \mathbb{R}^n \tag{2.14}$$

where $f(t, y) : I \times G \to \mathbb{R}^n$ is a continuous function.
In this respect, denote

$$z = (t, y) \in \mathbb{R}^{n+1}, D = I \times G \subseteq \mathbb{R}^{n+1}$$

and consider

$$g(z) : D \to \mathbb{R}^{n+1}$$

defined by

$$g(z) = coloumn(1, f(t, y)) \tag{2.15}$$

With these notations, the system (2.14) *can be written as an autonomous ODE*

$$\frac{dz}{dt} = g(z) \tag{2.16}$$

where $g \in \mathcal{C}^1(D; \mathbb{R}^{n+1})$ provided f is continuously differentiable on $(t, y) \in I \times G$. Using Proposition 2.0.6 we restate the conclusion of Theorem 2.0.9 as a result for linear PDE (2.7).

Proposition 2.0.11. *Let $a \in D \subseteq \mathbb{R}^n$ be fixed such that $f(a) \neq 0$, where $f \in \mathcal{C}^1(D; \mathbb{R}^n)$ defines the characteristic system* (2.1). *Let $\{u_i(y) : y \in V(a) \subseteq D\}, i \in \{1, ..., n-1\}$ be the $(n-1)$ first integral for* (2.1) *constructed in Theorem 2.0.8. Then an arbitrary solution $\{u(x) : x \in V(a) \subseteq D\}$ of the linear PDE* (2.7) *can berepresented*

$$u(x) = h(u_1(x), ..., u_{n-1}(x)), x \in V_1(a) \subseteq V(a)$$

where $h \in \mathcal{C}^1(\theta, \mathbb{R})$ and $\theta \subseteq \mathbb{R}^{n-1}$ is open set. In particular, each $\{u_i : x \in V(a) \subseteq D\}$ is a solution for the linear PDE (2.7), *$i \in \{1, ..., n-1\}$*

Remark 2.0.12. *A strategy based on the corresponding Cauchy characteristic system of ODE can be used for solving the qausilinear PDE given in* (2.6).

2.1 Cauchy Problem for the Hamilton-Jacobi Equations

A linear Hamiltonion-Jacobi (H-J) is defined by the following first order PDE

$$\partial_t S(t,x) + < \partial_x S(t,x), g(t,x) >= L(t,x), t \in I \subseteq \mathbb{R}, x \in G \subseteq \mathbb{R}^n \tag{2.17}$$

where $S \in \mathcal{C}^1(I \times G; \mathbb{R})$ is the unknown function

$$\partial_t S = \frac{\partial S}{\partial t}, \partial_x S = \frac{\partial S}{\partial x}$$

and

$$g(t,x) : I \times G \to \mathbb{R}^n, L(t,x) : I \times G \to \mathbb{R}$$

are given continuously differentiable functions. Here $I \subseteq \mathbb{R}$ and $G \subseteq \mathbb{R}^n$ are open sets and for each $t_0 \in I, h \in \mathcal{C}^1(G;\mathbb{R})$ fixed. The following Cauchy problem can be defined. Find a solution $S(t,x) : (t_0-\alpha, t_0+\alpha) \times D \to \mathbb{R}$ verifying (H-J) equation (2.17) $(t,x) \in (t_0-\alpha, t_0+\alpha) \times D, D(open) \subseteq G$ such that $S(t_0,x) = h(x), x \in D$, where $t_0 \in I$ and $h \in \mathcal{C}^1(G;\mathbb{R})$ are fixed, and $(t_0-\alpha, t_0+\alpha) \subseteq I$. A causilinear Hamiltonion-Jacobi (H-J) equation is defined by the following first order PDE

$$\partial_t S(t,x) + < \partial_x S(t,x), g(t,x,S(t,x)) >= L(t,x,S(t,x)), t \in I \subseteq \mathbb{R}, x \in G \subseteq \mathbb{R}^n \tag{2.18}$$

where $g \in \mathcal{C}^1(I \times G \times \mathbb{R}; \mathbb{R}^n), L \in \mathcal{C}^1(I \times G \times \mathbb{R}; \mathbb{R})$ are fixed and $I \subseteq \mathbb{R}, G \in \mathbb{R}^n$ are open sets. A Cauchy problem for causilinear (H-J)(2.18)is defined as follows:let $t_0 \in I$and $h \in \mathcal{C}^1(G;\mathbb{R})$ be given and find a solution $S(t,x) : (t_0-\alpha, t_0+\alpha) \times D \to \mathbb{R}$ verifying (2.18)for any $t \in (t_0-\alpha, t_0+\alpha) \subseteq I$ and $x \in D(open) \subseteq G$ such that $S(t_0,x) = h(x), x \in D$. A unique Cauchy problem solution of (2.17) and (2.18) are constructed using the Cauchy method of characteristics which relies on the corresponding Cauchy characteristic system of ODE. In this respect, we present this algorithm for (2.18) and let $\{F(t,\lambda) : t \in (t_0-\alpha, t_0+\alpha), \lambda \in \Sigma \subseteq G\}$ be the local flow generated by the following characteristic system associated with(2.18), $F(t,\lambda) = (\varphi(t,\lambda), u(t,\lambda)) \in \mathbb{R}^{n+1}$

$$\begin{cases} \frac{dF}{dt}(t,\lambda) = & f(t, F(t,\lambda)), (t,\lambda) \in (t_0-\alpha, t_0+\alpha) \times \Sigma \\ F(t_0,\lambda) = & (\lambda, u(t_0,\lambda)) = (,\lambda, h(\lambda)), \lambda \in \Sigma \subseteq G \end{cases} \tag{2.19}$$

where $f(t,x,u) = (g(t,x,u), L(t,x,u)) \in \mathbb{R}^{n+1}$ and $h \in \mathcal{C}^1(G;\mathbb{R})$ is fixed. Using $det\left(\frac{\partial\varphi}{\partial\lambda}(t,\lambda) \right) \neq 0$ for any $t \in (t_0-\alpha, t_0+\alpha)$ and $\lambda \in \Sigma(compact) \subseteq G$ (see $\frac{\partial\varphi}{\partial\lambda}(t_0,\lambda) = I_n$ and we may assume that $\alpha > 0$ is sufficiently small) an implicit function theorem can be applied to the algebraic equation

$$\varphi(t,\lambda) = x \in V(x_0) \subseteq G \tag{2.20}$$

We find the unique solution of (2.20)

$$\lambda = (\psi_1(t,x), ..., \psi_n(t,x)) : (t_0-\alpha, t_0+\alpha) \times V(x_0) \to \Sigma \tag{2.21}$$

such that

$$\psi_i \in \mathcal{C}^1(t_0-\alpha, t_0+\alpha) \times V(x_0);\mathbb{R}), i \in \{1,...,n\}$$

fulfills

$$\varphi(t, \psi_1(t,x), ..., \psi_n(t,x)) = x \in V(x_0) \subseteq G \tag{2.22}$$

for any $t \in (t_0-\alpha, t_0+\alpha)$ and $x \in V(x_0)$

$$\psi_i(t_0, x) = x_i, i = 1, ..., n, \psi = (\psi_1, ..., \psi_n) \tag{2.23}$$

w Define $S(t,x) = u(t, \psi(t,x))$ and it will be a continuously differentiable function on $(t_0-\alpha, t_0+\alpha) \times V(x_0)$ satisfying

$$S(t_0, x) = u(t_0, \psi(t_0, x)) = u(t_0, x) = h(x), x \in V(x_0) \tag{2.24}$$

In addition using $\psi(t, \varphi(t, \lambda)) = \lambda, t \in (t_0-\alpha, t_0+\alpha)$, and taking the derivative with respect to the variable t, we get $S(t, \varphi(t,\lambda)) = u(t,\lambda)$ and

$$\partial_t S(t, \varphi(t,\lambda)) + < \partial_x S(t, \varphi(t,\lambda)), g(t, \varphi(t,\lambda), u(t,\lambda)) >= L(t, \varphi(t,\lambda), u(t,\lambda)) \tag{2.25}$$

$\forall t \in (t_0-\alpha, t_0+\alpha), \lambda \in V_1(x_0)$ where $V_1(x_0) \subseteq V(x_0)$ is taken such that $\varphi(t,\lambda) \in V(x_0)$ if $(t,\lambda) \in (t_0-\alpha, t_0+\alpha) \times V_1(x_0)$. In particular, for $\lambda = \psi(t,x)$ into (1.25), we obtain

$$\partial_t S(t,x) + < \partial_x S(t,x), g(t,x,S(t,x)) >= L(t,x,S(t,x)) \tag{2.26}$$

where $\varphi(t, \psi(t,x)) = x$ and $u(t, \psi(t,x)) = S(t,x)$ are used. In addition,the Cauchy problem solution for (2.18) is unique and assuming that another solution $\{v(t,x)\}$ of (2.18) satisfies $v(t_0,x) = h(x)$for $x \in D \subseteq G$ then $v(t,x) = S(t,x), \forall t \in (t_0 - \widetilde{\alpha}, t_0 + \widetilde{\alpha}), x \in V(x_0) \bigcap D$ where $\widetilde{\alpha} > 0$. It relies on the unique Cauchy problem associated withODE (2.19). The proof is complete.

2.2 Nonlinear First Order PDE

2.2.1 Examples of Scalar NonlinearODE

We consider a simple scalar equation given implicitly by

$$F(x, y(x), y'(x)) = 0, y'(x) = \frac{dy}{dx}(x) \tag{2.27}$$

where $F(x,y,z) : D(open) \subseteq \mathbb{R}^3 \to \mathbb{R}$ be second order continuously differentiable satisfying

$$F(x_0, y_0, z_0) = 0, \partial_z F(x_0, y_0, z_0) \neq 0 \tag{2.28}$$

for some $(x_0, y_0, z_0) \in D$ fixed. Notice that for a smooth curve

$$\{\gamma(t) = (x(t), y(t), z(t))) \in D : t \in [0, a]\}$$

with

$$x(0) = x_0, y(0) = y_0, z(0) = z_0$$

we get

$$F(\gamma(t)) = 0, t \in [0, a] \tag{2.29}$$

provided

$$0 = \frac{d}{dt}[F(\gamma(t))] = \partial_x F(\gamma(t))\frac{dx}{dt}(t) + \partial_y F(\gamma(t))\frac{dy}{dt}(t) + \partial_z F(\gamma(t))\frac{dz}{dt}(t), t \in [0, a] \tag{2.30}$$

Using (2.30) we may and do define a corresponding characteristic system associated with (2.27)

$$\left\{ \begin{array}{ll} \frac{dx}{dt} = & \frac{\partial F}{\partial z}(x, y, z), \frac{dy}{dt} = z\frac{\partial F}{\partial z}(x, y, z) \\ \frac{dz}{dt} = & -[\frac{\partial F}{\partial x}(x, y, z) + \frac{\partial F}{\partial y}(x, y, z)z], x(0) = x_0, y(0) = y_0, z(0) = z_0 \end{array} \right. \tag{2.31}$$

Notice that each solution of (2.31) satisfies (2.29). Define $x = x(t), y = y(t), z = z(t), t \in [-a, a]$, the unique Cauchy problem solution of (2.31) and by definition $\frac{dx}{dt}(0) = \partial_z F(x_0, y_0, z_0) \neq 0$ allows to apply an implicit function theorem for solving the following scalar equation

$$x(t) = x \in V(x_0) \subseteq I(\text{interval}) \tag{2.32}$$

We find a unique continuously derivable function $t = \widetilde{\tau}(x) : V(x_0) \to (-a, a)$

$$x(\widetilde{\tau}(x)) = x \,\text{and}\, \widetilde{\tau}(x(t)) = t(\frac{d\widetilde{\tau}}{dx}(x)\frac{dx}{dt}(\widetilde{\tau}(x)) = 1) \tag{2.33}$$

Denote

$$\widehat{y}(x) = y(\widehat{\tau}(x)), \widehat{z}(x) = z(\widehat{\tau}(x)), x \in V(x_0) \subseteq I$$

and it is easily seen that

$$\frac{d\widehat{y}}{dx} = \frac{dy}{dt}(\widehat{\tau}(x)).\frac{d\widehat{\tau}}{dx}(x) = \widehat{z}(x)$$

It implies that

$$F(x, \widehat{y}(x), \widehat{y'}(x)) = 0, \forall x \in V(x_0)$$

provided

$$F(x(t), y(y), z(t)) = 0, t \in (-a, a)$$

is used. It shows that $\{\widehat{y}(x) : x \in V(x_0)\}$ is a solution of the scalar nonlinear differential equation (2.27).

Remark 2.2.1. *In getting the characteristic system* (2.31)*we must confine ourselves*

to the following constraints $y(t) = \widehat{y}(x(t)), \frac{dy}{dt}(t) = \frac{d\widehat{y}}{dx}(x(t)).\frac{dx}{dt}(t) = z(t)\frac{dx}{dt}$ *where* $z(t) = \frac{d\widehat{y}}{dx}(x(t))$ *and* $\{\widehat{y}(x), x \in I \subseteq \mathbb{R}\}$ *is a solution of* (2.27). *The following two examples can be solved using the algorithm of the characteristic system used for the scalar equation*(2.27).

Example 2.2.1. *(Clairant and Lagrange equations)*

$$\begin{cases} y = & xa(y') + b(y') \ \text{(Clairaut equation)} \\ y = & xy' + b(y') \ \text{(Lagrange equation,a(z)=z)} \end{cases} \tag{2.34}$$

Here $F(x, y, z) = xa(z) + b(z) - y$ *and the corresponding characteristic system is given by*

$$\begin{cases} \frac{dx}{dt} = & xa'(z) + b'(z), \frac{dz}{dt} = -a(z) + z, \frac{dz}{dt} = z(xa'(z) + b'(z)) \\ x(0) = & x_0 z(0) = z_0 y(0) = y_0 \end{cases} \tag{2.35}$$

where

$$x_0 a(z_0) + b(z_0) - y_0$$

and

$$x_0 a'(z_0) + b'(z_0) \neq 0$$

Example 2.2.2. *(Total differential equations)*

$$\frac{dy}{dx} = \frac{g(x, y)}{h(x, y)} \tag{2.36}$$

where $g, h : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ *are continuously differentiable functions and* $h(x, y) \neq 0, \forall (x, y) \in D$. *Formally,* (2.36) *can be written as*

$$-g(x, y)dx + h(x, y)dy = 0 \tag{2.37}$$

and (2.36) *is a total differential equation if a second order continuously differentiable function* $F : D \to \mathbb{R}$ *exists such that*

$$\frac{\partial F}{\partial x}(x, y) = -g(x, y), \frac{\partial F}{\partial y}(x, y) = h(x, y) \neq 0 \tag{2.38}$$

If $y = y(x), x \in I$, *is a solution of* (2.36) *then* $F(x, y(x)) = constant, x \in I$, *provided* F *fulfils* (2.38). *In conclusion,assuming*(2.38), *the nonlinear first order equation* (2.36) *is solved provided the corresponding algebraic equation*

$$F(x, y(x)) = c \tag{2.39}$$

is satisfied, where the constant c *is parameter.*

2.2.2 Nonlinear Hamilton-Jacobi E quations

A Hamilton-Jacobi equation is a first order PDE of the following form

$$\partial_t u(t,x) + H(t,x,u(t,x),\partial_x u(t,x)) = 0, t \in I \subseteq \mathbb{R}, x \in D \subseteq \mathbb{R}^n \tag{2.40}$$

where $H(t,x,u,p) : I \times D \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a second order continuously differentiable function.

Definition 2.2.2. *A solution for H-J equation* (2.40) *means a first order continuously differentiable function* $u(t,x) : I_a(x_0) \times B(x_0,\rho)$ *where* $B(x_0,\rho) \subseteq I$*is a ball centered at* $x_0 \in D$ *and* $I_a(x_0) = (x_0 - a, x_0 + a) \subseteq I$. *A Cauchy problem for the H-J equation*(2.40) *means to find a solution* $u \in \mathcal{C}^1(I_a(x_0) \times B(x_0,\rho))$ *of* (2.40) *such that* $u(t_0,x) = u_0(x), x \in B(x_0,\rho)$ *where* $t_0 \in I$ *and* $u_0 \in \mathcal{C}^1(D;\mathbb{R})$ *are fixed.*

solution for Cauchy problem associated with H-J equation (2.40) is found using the corresponding characteristic system

$$\begin{cases} \frac{dx}{dt} = & \partial_p H(t,x,u,p), x(t_0) = \xi \in B(x_0,\rho) \subseteq D \\ \frac{dp}{dt} = & -[\partial_x H(t,x,u,p) + p\partial_u H(t,x,u,p)], p(t_0) = p_0(\xi) \\ \frac{du}{dt} = & -H(t,x,u,p) + < p, \partial_p H(t,x,u,p) >, u(t_0) = u_0(\xi) \end{cases} \tag{2.41}$$

where $p_0(\xi) = \partial_\xi u_0(\xi)$ and $u_0 \in \mathcal{C}^1(D;\mathbb{R})$ are fixed. Consider that

$$\{(\widehat{x}(t,\xi)), \widehat{p}(t,\xi), \widehat{u}(t,\xi) : t \in I_a(x_0), \xi \in B(x_0,\rho)\} \tag{2.42}$$

is the unique solution fulfilling ODE (2.40). By definition $\partial_\xi \widehat{x}(0,\xi) = I_n$ and assuming that $a > 0$ is sufficiently small, we admit

$$\partial_\xi \widehat{x}(t,\xi) \, \text{is nonsingular for any} (t,x) \in I_a(x_0) \times B(x_0,\rho) \tag{2.43}$$

Using (2.47) we may and do apply the standard implicit functions theorem for solving the algebraic equation

$$\widehat{x}(t,\xi) = x \in B(x_0,\rho_1), t \in I_\alpha(x_0) \tag{2.44}$$

We get a continuously differentiable mapping

$$\xi = \psi(t,x) : I_\alpha(x_0) \times B(x_0,\rho_1) \to B(x_0,\rho)$$

such that

$$\begin{array}{rcll} \widehat{x}(t,\psi(t,x)) & = & x, \psi(t_0,x) = x & \text{and} \\ \psi(t,\widehat{x}(t,\xi)) & = & \xi & \text{for any} \end{array}$$

$$t \in I_\alpha(x_0), x \in B(x_0,\rho_1) \tag{2.45}$$

Define the following continuously differentiable function

$$u \in \mathcal{C}^1(I_\alpha(x_0) \times B(x_0, \rho_1); \mathbb{R}), p \in \mathcal{C}^1(I_\alpha(x_0) \times B(x_0, \rho_1); \mathbb{R}^n)$$

$$u(t, x) = \widehat{u}(t, \psi(t, x)), p(t, x) = \widehat{p}(t, \psi(t, x)) \tag{2.46}$$

By definition $u(t_0, x) = u_0(x), x \in B(x_0, \rho_1) \subseteq D$ and to show that $\{u(t, x) : t \in I_\alpha(x_0), x \in B(x_0, \rho_1)\}$ is a solution for (2.40) satisfying $u(t_0, x) = u_0(x), x \in B(x_0, \rho_1) \subseteq D$, we need to show

$$\begin{cases} \partial_t u(t, x) = & -H(t, x, u(t, x), p(t, x)) \\ p(t, x) = & \partial_x u(t, x), (t, x) \in I_\alpha(x_0) \times B(x_0, \rho_1) \end{cases} \tag{2.47}$$

The second equation of (2.47)is valid if

$$\partial_\xi \widehat{u}(t, \xi) = \widehat{p}(t, \xi) \partial_\xi \widehat{x}(t, \xi), (\widehat{p} \in \mathbb{R}^n)$$

is a row vector holds for each

$$t \in I_\alpha(x_0), \xi \in B(x_0, \rho) \tag{2.48}$$

which will be proved in the second form

$$\partial_k \widehat{u}(t, \xi) =< \widehat{p}(t, \xi), \partial_k \widehat{x}(t, \xi) >, k \in \{1, ..., n\} \tag{2.49}$$

where

$$\partial_k \varphi(y, \xi) = \frac{\partial \varphi(t, \xi)}{\partial \xi_k}, \xi = (\xi_1, ..., \xi_n)$$

Using the characteristic system (2.40) we notice that

$$\lambda_k(t, \xi) = \partial_k \widehat{u}(t, \xi) - < \widehat{p}(t, \xi), \partial_k \widehat{x}(t, \xi) >$$

fulfils

$$\partial_k(t_0, \xi) = \partial_k u_0(\xi) - < \partial_\xi u_0(\xi), e_k >= 0, k \in \{1, ..., n\} \tag{2.50}$$

where

$$\{e_1, ..., e_n\} \subseteq \mathbb{R}^n$$

is the canonical basis. In addition, by a direct computation, we obtain

$$\begin{aligned} \frac{d\lambda_k}{dt}(t, \xi) \quad &= \quad \partial_k[\frac{d\widehat{u}}{dt}(t, \xi)] - < \frac{d\widehat{p}(t, \xi)}{dt} \\ \partial_k \widehat{x}(t, \xi) \quad &> - < \quad \widehat{p}(t, \xi), \partial_k[\frac{d\widehat{x}}{dt}(t, \xi)] >, t \in I_\alpha(x_0) \end{aligned} \tag{2.51}$$

Notice that

$$\begin{aligned}\partial_k[\frac{d\widehat{u}}{dt}(t,\xi)] &= -\partial_k[H(t,\widehat{x}(t,\xi),\widehat{u}(t,\xi),\widehat{p}(t,\xi))]+<\partial_k\widehat{p}(t,\xi)\\ \frac{\widehat{x}}{dt}(t,\xi) &>+< \widehat{p}(t,\xi),\partial_k[\frac{d\widehat{x}}{dt}(t,\xi)]> \end{aligned} \tag{2.52}$$

and

$$\frac{d\widehat{p}}{dt}(t,\xi)=-[\partial_x H(t,\widehat{x}(t,\xi)),\widehat{u}(t,\xi),\widehat{p}(t,\xi_1)+\widehat{p}(t,\xi)\partial_u H(t,\widehat{x}(t,\xi)),\widehat{u}(t,\xi),\widehat{p}(t,\xi)] \tag{2.53}$$

Combining (2.52)and (2.53) we obtain

$$\begin{aligned}\frac{d\lambda_k}{dt}(t,\xi) &= -\partial_u H(t,\widehat{x}(t,\xi),\widehat{u}(t,\xi),\widehat{p}(t,\xi)).\partial_k\widehat{u}(t,\xi)+\partial_u H(t,\widehat{x}(t,\xi),\widehat{u}(t,\xi),\widehat{p}(t,\xi))\\ &< \widehat{p}(t,\xi),\partial_k\widehat{x}(t,\xi)>=-\partial_u H(t,\widehat{x}(t,\xi),\widehat{u}(t,\xi),\widehat{p}(t,\xi)).\lambda_k(t,\xi)\end{aligned} \tag{2.54}$$

which is a linear scalar equation for the unknown$\{\lambda_k\}$ satisfying $\lambda_k(t_0,\xi)=0, k\in\{1,...,n\}$(see(2.51)). It follows $\lambda_k(t,\xi)=0,\forall(t,\xi)\in I_\alpha(x_0)\times B(x_0,\rho_1)$ and for any $k\in\{1,...,n\}$, which proves

$$p(t,x)=\partial_x u(t,x),(t,x)\in I_\alpha(x_0)\times B(x_0,\rho_1) \tag{2.55}$$

standing for the second equation in(2.47). Using(2.55), $\widehat{u}(t,\xi)=u(t,\widehat{x}(t,\xi))$and the third equation of the characteristic system (2.40) we see easily that

$$\begin{aligned}\partial_t u(t,\widehat{x}(t,\xi))+<\widehat{p}(t,\xi),\frac{d\widehat{x}}{dt}(t,\xi) &>=- H(t,\widehat{x}(t,\xi),\widehat{u}(t,\xi),\widehat{p}(t,\xi))\\ &+< \widehat{p}(t,\xi),\frac{d\widehat{x}}{dt}(t,\xi)>\end{aligned} \tag{2.56}$$

which lead us to the first equation of (2.47)

$$\partial_t u(t,\widehat{x}(t,\xi))=-H(t,\widehat{x}(t,\xi),\widehat{u}(t,\xi),\widehat{p}(t,\xi)) \tag{2.57}$$

for any $(t,\xi)\in I_\alpha(x_0)\times B(x_0,\rho)$. In particular ,taking $\xi=\psi(t,x)$ in (2.58), we get

$$\partial_t u(t,x)=-H(t,x,u(t,x),\partial_x u(t,x)),u(t_0,x)=u_0(x) \tag{2.58}$$

$$\forall(t,x)\in(t_0-\alpha,t_0+\alpha)\times B(x_0,\rho_1)$$

which stands for the existence of a Cauchy problem solution. The uniqueness of the Cauchy problem solution for H-J equation(2.40) can be easily proved using the fact that any other solution $\widehat{u}(t,x),(t,x)\in I_\alpha(x_0)\times B(x_0,\rho_2)$ satisfying (2.49) and $\widehat{u}(t_0,x)=u_0(x),x\in B(x_0,\rho_2)$ induces a solution of the same ODE (2.40). The conclusion is that the uniqueness property for the Cauchy problem solution of ODE (2.41) implies that the H-J equation (2.40) has a unique Cauchy problem solution. The above given computations and considerations regarding the H-J equation (2.40) will be stated as

Proposition 2.2.3. *Let $H(t,x,u,p):I\times D\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ be a second order*

continuously differentiable scalar function, where $I \subseteq \mathbb{R}, D \subseteq \mathbb{R}^n$ are open sets. Let $(t_0, x_0) \in I \times D$and $u_0 \in \mathcal{C}^1(D; \mathbb{R})$ are fixed. Then the following nonlinear H-J equation

$$\partial_t u(t,x) + H(t,x,u(t,x),\partial_x u(t,x)) = 0 \tag{2.59}$$

This has a unique Cauchy problem solution satisfying

$$u(t_0,x) - u_0(x), x \in B(x_0,\rho_0) \subseteq D$$

In addition,the unique Cauchy problem solution

$$\{u(t,x) : t \in (t_0 - \alpha, t_0 + \alpha), x \in B(x_0,\rho_0)\}$$

of (2.59) *is defined by $u(t,x) = \widehat{u}(t,\psi(t,x))$where $\{\varphi(t,x) : t \in I_\alpha(x_0), x \in B(x_0,\rho_0)\}$ is the unique solution of the algebraic* (2.44) *and*

$$\{(\widehat{x}(t,\xi), \widehat{p}(t,\xi), \widehat{u}(t,\xi)) : t \in I_\alpha(x_0), \xi \in B(x_0,\rho)\}$$

is the unique solution of the characteristic system (2.41)

Starting with the H-J equation (2.59) (see(2.40)) we may associate the following system of H-J equation for the unknown $p(t,x) = \partial_x u(t,x) \in \mathbb{R}^n$

$$\begin{aligned} \partial_t p(t,x) \quad &+ < \quad \partial_p H(t,x,u(t,x),p(t,x)) \partial_x p(t,x) > \\ &= - \quad [\partial_x H(t,x,u(t,x),p(t,x)) + p(t,x)\partial_u H(t,x,u(t,x),p(t,x))] \end{aligned} \tag{2.60}$$

where $p = (p_1, ..., p_n), \partial_p H$ and $\partial_x H$ are row vectors. Using $\partial_{x_i} p(t,x) = \partial_x p_i(t,x), i \in \{1, ..., n\}$ we notice that (2.60) can be written as follows

$$\begin{aligned} \partial_t p_i(t,x) \quad &+ < \quad \partial_x p_i(t,x), \partial_p H(t,x,u(t.x),p(t,x)) > \\ &= - \quad [\partial_{x_i} H(t,x,u(t,x),p(t,x)) + p_i(t,x)\partial_u H(t,x,u(t,x),p(t,x))] \end{aligned} \tag{2.61}$$

for each $i \in \{1, ..., n\}$, and

$$\{(\partial_x(t,\xi)), \partial_p(t,\xi) = p(t,\partial_x(t,\xi)), \partial_u(t,\xi) = u(t,\partial_x(t,\xi) : t \in I_\alpha(x_0))\} \tag{2.62}$$

satisfies the characteristic system (2.42) provided

$$\begin{cases} \frac{d\widehat{x}(t,\xi)}{dt} = & \partial_p H(t,\widehat{x}(t,\xi),\widehat{u}(t,\xi),\widehat{p}(t,\xi)), t \in I_\alpha(x_0) \\ \widehat{x}(t_0,\xi) = & \xi \end{cases} \tag{2.63}$$

The additional system of H-J equation (2.61) stands for a causilinear system of evolution equations

$$\begin{cases} \partial_t v_i(t,x) + < \partial_x v_i(t,x), X(t,x,v(t,x)) >= & L_i(t,x,v(t,x)) \\ v_i(t_0,x) = v_6^0, i \in \{1, ...n\} \end{cases} \tag{2.64}$$

Which allows to use the corresponding characteristic system suitable for a scalar equation. It relies on the unique vector field $X(t,x,v)$ deriving each scalar equation

in the system (2.64).

Remark 2.2.4. *In the case that the unique vector field $X(t,x,v)$ is replaced by some $X_i(t,x,v)$, for each $i \in \{1,...,n\}$, which are not commuting with respect to the Lie bracket $[X_i(t,x,.), X_i(t,x,.)](v) \neq o$, for some $i \neq j$, then the integration of the system* (2.64)*changes drastically.*

2.2.3 Exercises

(E_1).Using the characteristic system method, solve the following Cauchy problems

$$\partial_t u(t,x) = (\partial_x u(t,x))^2, u(0,x) = cosx, x \in \mathbb{R}, u \in \mathbb{R} \tag{2.65}$$

$$\begin{cases} \partial_t u(t,x_1,x_2) = & x_1\partial_{x_1} u(t,x_1,x_2) + (\partial_{x_2} u(t,x_1,x_2))^2 \\ u(0,x_1,x_2) = & x_1 + x_2, (x_1,x_2) \in \mathbb{R}^2, u \in \mathbb{R} \end{cases} \tag{2.66}$$

(E_2). Let $f(t,x) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and $\varphi(x) : \mathbb{R} \to \mathbb{R}$ be some first order continuously differentiable functions. Find the Cauchy problem solution of the following linear H-J equations

$$\begin{cases} \partial_t u(t,x) + & \Sigma_{i=1}^n a_i(t)\partial_{x_i} u(t,x) = f(t,x) \\ u(0,x) = & \varphi(x) \end{cases} \tag{2.67}$$

where $a(t) = (a_1(t),...,a_n(t)) : \mathbb{R} \to \mathbb{R}^n$ is a continuous function.

$$\begin{cases} \partial_t u(t,x) + < A(t)x.\partial_x u(t,x) >= f(t,x) \\ u(0,x) = \varphi(x) \end{cases} \tag{2.68}$$

where $A(t) : \mathbb{R} \to M_{n\times n}$ is a continuous mapping.

2.3 Stationary Solutions for Nonlinear First Order PDE

2.3.1 Introduction

We consider a nonlinear equation

$$H_0(x, \partial_x u(x), u(x)) = constant\, \forall x \in D \subseteq \mathbb{R}^n \tag{2.69}$$

where

$$H_0(x,p,u) : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$$

is a scalar continuously differentiable function

$$H_0 \in \mathcal{C}^1(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R})$$

and $\partial_x u = (\partial_1 u, ..., \partial_n u)$ stands for the gradient of a scalar function. A standard solution for (2.69) means to find $u(x) : B(x_0, \rho) \subseteq D \to \mathbb{R}, u \in \mathcal{C}^1(B(x_0, \rho))$ such that $H_0(x, \partial_x u(x), u(x)) = \text{constant} \,\forall\, x \in B(x_0, \rho) \subseteq D$. The usual method of solving (2.69) uses the associated characteristic system

$$\frac{d\widehat{z}}{dt} = Z_0(\widehat{z}), \widehat{z}(0, \lambda) = \widehat{z}(\lambda), \lambda \in \Lambda \subseteq \mathbb{R}^{n-1}, t \in (-a, a) \tag{2.70}$$

where $Z_0(z) : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}, z = (x, p, u)$, is the characteristic vector field corresponding to $H_0(z)$

$$Z_0(z) = (X_0(z), P_0(z), U_0(z)), X_0(z) = \partial_p H_0(z) \in \mathbb{R}^n \tag{2.71}$$

$$U_0(z) =< p, X_0(z) >, P_0(z) = -(\partial_x H_0(z) + p\partial_u H_0(z)) \in \mathbb{R}^n$$

For a fixed Cauchy condition $\widehat{z}(\lambda) = (\widehat{x}(\lambda), \widehat{p}(\lambda), \widehat{u}(\lambda))$ given on a domain $\lambda \in \Lambda \subseteq \mathbb{R}^{n-1}$, a compatibility condition

$$\partial_i \widehat{u}(\lambda) =< \widehat{p}(\lambda), \partial_i \widehat{x}(\lambda) >, i \in \{1, ..., n-1\} \tag{2.72}$$

is necessary. In addition, both vector fields $Z_0(z)$ and the parametrization $\{\widehat{z}(\lambda) : \lambda \in \Lambda\}$ must satisfy a nonsingularity condition

$$\begin{gathered}\text{the vectors in } \mathbb{R}^n, X_0(\widehat{z}(\lambda)), \partial_1 \widehat{x}(\lambda), ..., \partial_{n-1} \widehat{x}(\lambda) \\ \text{are linearly independent for any} \lambda \in \Lambda \in \mathbb{R}^{n-1}\end{gathered} \tag{2.73}$$

By definition, the characteristic vector field $\{\partial_z H_0(z) : z \in \mathbb{R}2n+1\}$ and we get

$$0 = \begin{cases} < \partial_z H_0(z), Z_0(z) > \\ < \partial_x H_0(z), X_0(z) > + < \partial_p H_0(z), P_0(z) > + \partial_u H_0(z) U_0(z) \end{cases} \tag{2.74}$$

for any $z = (x, p, u) \in \mathbb{R}^{2n+1}$. Using (2.74) for the local solution $\{\widehat{z}(t, \lambda) : t \in (-a, a)\}$ of the characteristic system (2.70) we obtain $H_0(\widehat{z}(t, \lambda)) = H_0(\widehat{z}(\lambda)), t \in (-a, a)$, for each $\lambda \in \Lambda \subseteq \mathbb{R}^{n-1}$. Looking for a stationary solution of the equation (2.69) we need to impose the following constaint

$$H_0(\widehat{z}(t, \lambda)) = H_0(z_0) \text{for any} (t, \lambda) \in (-a, a) \times \Lambda \tag{2.75}$$

where $z_0 = \widehat{z}(0) = (x_0, p_0, u_0)(0 \in int\Lambda$ for simplicity). The condition (2.73) allows to apply the standard implicit function theorem and to solve the algebraic equation

$$\widehat{x}(t, \lambda) = \lambda \in B(x_0, \rho) \subseteq D \tag{2.76}$$

We get smooth functions $t = \tau(x) \in (-a, a)$ and $\lambda = \psi(x) \in \Lambda$ such that

$$\widehat{x}(\tau(x), \psi(x)) = x \in B(x_0, \rho) \subseteq D, \tau(x_0) = 0, \psi(x_0) = 0 \in \Lambda \tag{2.77}$$

A solution for the nonlinear equation (2.69) is obtained as follows

$$u(x) = \widehat{u}(\tau(x), \psi(x)), p(x) = \widehat{p}(\tau(x), \psi(x)), x \in B(x_0, \rho) \subseteq \mathbb{R}^n \tag{2.78}$$

where $p(x)$ fulfils

$$p(x) = \partial_x u(x)$$

Obstruction. We need explicit condition to compute $\{\widehat{z}(\lambda) : \lambda \in \Lambda\}$ such that (2.72), (2.73) and (2.75) are verified.

2.3.2 The Lie Algebra of Characteristic Fields

Denote $\mathcal{H} = \mathcal{C}^\infty(\mathbb{R}^{2n+1}; \mathbb{R})$ the space consisting of the scalar functions $H(x, p, u) : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ which are differentiable of any order. For each pair $H_1, H_2 \in$ define the Poisson bracket

$$\{H_1, H_2\}(z) =< \partial_z H_2(z), Z_1(z) >, z = (x, p, u) \in \mathbb{R}^{2n+1} \tag{2.79}$$

where $\partial_z H_2(z)$ stands for the gradient of a scalar function $H_2 \in \mathcal{H}$ and $Z_1(z) = (X_1(z), P_1(z), U_1(z)) \in \mathbb{R}^{2n+1}$ is the characteristic field corresponding to $H_1 \in \mathcal{H}$. We recall that Z_1 is obtained from $H_1 \in \mathcal{H}$ such that the following equations

$$X_1(z) = \partial_p H_1(z), P_1(z) = -(\partial_x H_1(z) + p\partial_u H_1(z)), U_1(z) =< p, \partial_p H_1(z) > \tag{2.80}$$

are satisfied. The linear mapping connecting an arbitrary $H \in \mathcal{H}$ and its characteristic field can be represented by

$$Z_H(z) = T(p)(\partial_z H)(z),\ z = (x, p, u) \in \mathbb{R}^{2n+1} \tag{2.81}$$

where the real $(2n+1) \times (2n+1)$ matrix $T(p)$ is defined by

$$T(p) = \begin{pmatrix} O & I_n & \theta \\ -I_n & O & -p \\ \theta^* & p^* & 0 \end{pmatrix} \tag{2.82}$$

where O-zero matrix of $M_{n\times n}$, $I_n$unity matrix of $M_{n\times n}$and $\theta \in \mathbb{R}^n$ is the null column vector. We notice that $T(p)$ is a skew symmetric matrix

$$[T(p)]^* = -T(p) \tag{2.83}$$

and as a consequence, the Poisson bracket satisfies a skew symmetric property

$$\{H_1, H_2\}(z) = \begin{cases} < \partial_z H_1(z), Z_2(z) > \\ < \partial_z H_1(z), T(p)\partial_z H_2(z) > \\ < [T(p)]^* \partial_z H_1(z), \partial_z H_2(z) > \\ -\{H_2, H_1\} \end{cases} \tag{2.84}$$

In addition, the linear space of characteristic fields $K \subseteq \mathcal{C}^{\infty}(\mathbb{R}^{2n+1}, \mathbb{R}^{2n+1})$ is the image of a linear mapping $S : D\mathcal{H} \to K$ where $D\mathcal{H} = \{\partial_z H : H \in \mathcal{H}\}$. In this respect, using (2.81) we define

$$S(\partial_z H)(z) = T(p)(\partial_z H)(z), z \in \mathbb{R}^{2n+1} \tag{2.85}$$

where the matrix $T(p)$ is given in (2.82). The linear space of characteristic fields $K = S(d\mathcal{H})$ is extended to a Lie algebra

$$L_k \subseteq \mathcal{C}^{\infty}(\mathbb{R}^{2n+1}, \mathbb{R}^{2n+1})$$

using the standard Lie bracket of vector fields

$$[Z_1, Z_2] = [\partial_z Z_2(z)]Z_1 - [\partial_z Z_1]Z_2(z),\ Z_i \in K, i = 1, 2 \tag{2.86}$$

On the other hand, each $H \in \mathcal{H}$ is associated with a linear mapping

$$\overrightarrow{H}(\varphi)(z) = \{H_1\varphi\}(z) =< \partial_z\varphi(z), Z_H(z)) >, z \in \mathbb{R}^{2n+1} \tag{2.87}$$

for each $\varphi \in \mathcal{H}$, where $Z_H \in K$ is the characteristic vector field corresponding to $H \in \mathcal{H}$ obtained from $\partial_z H$ by $Z_H(z) = T(p)(\partial_z H)(z)$ (see (2.81)). Define a linear space consisting of linear mappings

$$\overrightarrow{\mathcal{H}} = \{\overrightarrow{H} : H \in \mathcal{H}\} \tag{2.88}$$

and extend $\overrightarrow{\mathcal{H}}$ to a Lie algebra L_H using the Lie bracket of linear mappings

$$[\overrightarrow{H_1}, \overrightarrow{H_2}] = \overrightarrow{H_1} \circ \overrightarrow{H_2} - \overrightarrow{H_2} \circ \overrightarrow{H_1} \tag{2.89}$$

The link between the two Lie algebras L_K(extending K) and L_H(extending $\overrightarrow{\mathcal{H}}$) is given by a homomorphism of Lie algebras

$$A : L_H \to L_K \,\text{satisfying}\, A(\overrightarrow{\mathcal{H}}) = K \tag{2.90}$$

and

$$A([\overrightarrow{H_1}, \overrightarrow{H_2}]) = [Z_1, Z_2] \in L_K \text{where} Z_i = A(\overrightarrow{H_i}),\ i \in \{1, 2\} \tag{2.91}$$

Remark 2.3.1. *The Lie algebra $L_H(\supseteq \overrightarrow{\mathcal{H}})$ does not coincide with the linear space $\overrightarrow{\mathcal{H}}$ and as a consequence,the linear space $K \subseteq L_k$. It relies upon the fact the linear mapping $\{\overrightarrow{H_1, H_2}\}$ generated by the Poisson bracket $\{H_1, H_2\} \in \mathcal{H}$ does not coincide with the Lie bracket $[\overrightarrow{H_1}, \overrightarrow{H_1}]$ defined in* (2.89)*.*

Remark 2.3.2. *In the particular case when the equation* (2.69) *is replaced by $H_0(x, p) : \mathbb{R}^{2n} \to \mathbb{R}$ is continuously differentiable then the above given analysis will be restricted to the space $\mathcal{H} = \mathcal{C}^{\infty}(\mathbb{R}^{2n}; \mathbb{R})$. If it is the case then the corresponding linear mapping $S : D\mathcal{H} \to K$ is determined by a simplectic matrix $T \in M_{2n\times 2n}$*

$$T = \begin{pmatrix} O & I_n \\ -I_n & O \end{pmatrix}, D\mathcal{H} = \{\partial_z H : H \in \mathcal{C}^{\infty}(\mathbb{R}^{2n}; \mathbb{R})\} \tag{2.92}$$

In addition, the linear spaces $\overrightarrow{H}$ and $K \subseteq \mathcal{C}^\infty(\mathbb{R}^{2n};\mathbb{R}^{2n})$ coincide with their Lie algebra L_H and correspondingly L_K as the following direct computation shows

$$[Z_1, Z_2](z) = T\partial_z H_{12},\ z \in \mathbb{R}^{2n} \tag{2.93}$$

where

$$Z_i = T\partial_z H_i,\ i \in \{1,2\}$$

and

$$H_{12} = \{H_1, H_2\}(z) =< \partial_z H_2(z), Z_1(z) >$$

is the Poisson bracket associated with two scalar functions $H_1, H_2 \in \mathcal{H}$. We get

$$T\partial_z H_{12}(z) = \begin{cases} T[\partial_z^2 H_2(z)]Z_1(z) + T(\partial_z Z_1^*(z))\partial_z H_2 \\ [\partial_z Z_2(z)]Z_1(z) + T[\partial_z(\partial_z^* H_1^* T^*)]\partial_z H_2(z) \\ [\partial_{z_2}(z)]Z_1(z) - T(\partial_z^2 H_1(z))T\partial_z H_2(z) \\ [\partial_z Z_2(z)]Z_1(z) - [\partial_z Z_1(z)]Z_2(z) \end{cases} \tag{2.94}$$

and the conclusion $\{L_H = \overrightarrow{H}, L_K = K\}$ is proved.

2.3.3 Parameterized Stationary Solutions

Consider a nonlinear first order equation

$$H_0(x, p(x), u(x)) = \text{const}\ \forall\, x \in D \subseteq \mathbb{R}^n \tag{2.95}$$

With the same notations as in section 2.4.2, let $\mathcal{H} = \mathcal{C}^\infty(\mathbb{R}^{2n+1};\mathbb{R})\}$, $z = (x, p, u) \in \mathbb{R}^{2n+1}$ and define the skew-symmetric matrix

$$T(p) = \begin{pmatrix} O & I_n & \theta \\ -I_n & O & -p \\ \theta^* & p^* & 0 \end{pmatrix} ,p \in \mathbb{R}^n \tag{2.96}$$

By definition, the linear spaces K and $D\mathcal{H}$ are

$$K = S(D\mathcal{H}),\ D\mathcal{C} = \{\partial_z H : H \in \mathcal{H}\} \tag{2.97}$$

where the mapping $S : D\mathcal{H} \to K$ satisfies

$$S(\partial_z H)(z) = T(p)\partial_z H(z),\ z \in \mathbb{R}^{2n+1} \tag{2.98}$$

Let $z_0 = (x_0, p_0, u_0) \in \mathbb{R}^{2n+1}$ and the ball $B(x_0, 2\rho) \subseteq \mathbb{R}^{2n+1}$ be fixed. Assume that

$$\text{there exist}\, \{Z_1, ..., Z_m\} \subseteq K \tag{2.99}$$

such that the smooth vector fields

$$\{Z_1(z), ..., Z_m(z) : z \in B(z_0, 2\rho)\}$$

are in involution over $\mathcal{C}^\infty(B(z_0, 2\rho))$

$$\text{(the Lie bracket)}[Z_i, Z_j](z) = \sum_{k=1}^{m} \alpha_{ij}^k(z) Z_k(z),\ i, j \in \{1, ..., m\}$$

where

$$\alpha_{ij}^k \in \mathcal{C}^\infty(B(z_0, 2\rho))$$

A parameterized solution of the equation (2.95) is given by the following orbit

$$\widehat{z}(\lambda, z_0) = G_1(t_1) \circ ... \circ G_m(t_m)(z_0),\ \lambda = (t_1, ..., t_m) \in \Lambda = \prod_1^m [-a_i, a_i] \tag{2.100}$$

where

$$G_i(\tau)(y),\ y \in B(z_0, \rho),\ \tau \in [-a_i, a_i]$$

is the local flow generated by Z_i. Notice that

$$N_{z_0} = \{z \in B(z_0, 2\rho) : z = \widehat{z}(\lambda, z_0),\ \lambda \in \Lambda\} \tag{2.101}$$

is a smooth manifold and

$$\dim M_{z_0} = \dim L(Z_1, ..., Z_m)(z_0) = m \leqslant n\, if\, Z_1(z_0), ..., Z_m(z_0) \in \mathbb{R}^{2n+1} \tag{2.102}$$

are linearly independent.

Remark 2.3.3. *Under the conditions of the hypothesis* (2.99) *and using the algebraic representation of a gradient system associated with* $\{Z_1, ..., Z_m\} \subseteq K$ *a system* $\{q_1, ..., q_m\} \subseteq \mathcal{C}^\infty(\Lambda, \mathbb{R}^m)$ *exists such that*

$$q_1(\lambda), ..., q_m(\lambda) \in \mathbb{R}^m arelinearlyindependent \tag{2.103}$$

and $\partial_\lambda \widehat{z}(\lambda, z_0) q_i(\lambda) = Z_i(\widehat{z}(\lambda, z_0))$ *for any* $\lambda \in \Lambda,\ i \in \{1, ..., m\}$

Definition 2.3.4. *A parameterized solution associated with* (2.95)*is defined by orbit* (2.100)*provided the equality* $H_0(\widehat{z}(\lambda, z_0)) = H_0(z_0),\ \forall\, \lambda \in \Lambda$*, is satisfied.*

Remark 2.3.5. *The conclusion* (2.103) *does not depend on the manifold structure given in* (2.101)*. Using* (2.104)*, we rewrite the equation* $H_0(\widehat{z}(\lambda, z_0)) = H_0(z_0),\ \forall\, \lambda \in \Lambda$*, in the following equivalent form*

$$0 = \{H_i, H_0\}(\widehat{z}(\lambda.z_0)) =< \partial_z H_0(\widehat{z}(\lambda, z_0)), Z_i(\widehat{z}(\lambda, z_0)) >,\ \lambda \in \Lambda \tag{2.104}$$

for each $i \in \{1, ..., m\}$*, where* $\{H_i, H_0\}$ *is the Poisson bracket associated with* $H_i, H_0 \in \mathcal{C}^1(\mathbb{R}^{2n+1}, \mathbb{R})$*, and* $Z_i = S(\partial_z H_i),\ i \in \{1, ..., m\}$*. The equations* (2.104) *are directly computed from the scalar equation* $H_0(\widehat{z}(\lambda, z_0)) = H_0(z_0),\ \forall\, \lambda \in \Lambda$*, by taking the corresponding Lie derivatives where* $\{q_1, ..., q_m\}$ *from* (2.103) *are used. As a consequence, the orbit defined in* (2.100)*, under the conditions* (2.103) *and* (2.102)*, will determine a parameterized solution of* (2.104)*.*

Remark 2.3.6. *A classical solution for* (2.95) *can be deduced from a parameterized*

solution $\{\widehat{z}(\lambda, z_0) : \lambda \in \Lambda\}$ *if we take* $m = n$ *and assume*

$$\text{the matrix } [\partial_\lambda \widehat{x}(0, z_0)] \in M_{n\times n} \text{ is nonsingular} \tag{2.105}$$

where the components $(\widehat{x}(\lambda, z_0), \widehat{p}(\lambda, z_0), \widehat{u}(\lambda, z_0)) = \widehat{z}(\lambda, z_0)$ *define the parameterized solution* $\{\widehat{z}(\lambda, z_0)\}$.

Proposition 2.3.7. *Assume the orbit*$\{\widehat{z}(\lambda, z_0) : \lambda \in \Lambda\}$ *given in* (2.100) *is a parameterized solution of* (2.95) *such that the condition* (2.105) *is satisfied.Let* $\lambda = \psi(x) : S(x_0, \rho) \to int\Lambda$ *be the smooth mapping satisfying* $\widehat{x}(\psi(x), z_0) = x \in B(x_0, \rho)$, $\psi(x_0) = 0$. *Denote* $u(x) = \widehat{u}(\psi(x), z_0)$ *and* $p(x) = \widehat{p}(\psi(x), z_0)$.*Then*

$$p(x) = \partial_x u(x) \, and \, H_0(x, p(x), u(x)) = H_0(z_0), for\ any\, x \in B(x_0, \rho) \subseteq \mathbb{R}^n \tag{2.106}$$

where $p(x)$, $x \in B(x_0, \rho))$ *verifies the following cuasilinear system of first order equations*

$$\begin{aligned} &[\partial_x H_0(x, p(x), u(x)) + p(x)\partial_u H_0(x, p(x), u(x))] \\ &+[\partial_x p(x)]^* \partial_p H_0(x, p(x), u(x)) = 0, \; x \in B(x_0, \rho) \end{aligned} \tag{2.107}$$

Proof. The component $\{\widehat{x}(\lambda, z_0) : \lambda \in \Lambda\}$ fulfills the condition of the standard implicit functions theorem (see (2.105)) and, by definition, the matrix

$$\partial_\lambda \widehat{x}(0.z_0) = \| X_1(z_0, ..., X_n(z_0)) \|$$

is composed by first vector-component of $Z_i(z) = (X_i(z), P_i(z), U_i(z))$ $i \in \{1, ..., n\}$, where $\{Z_1(z), ..., Z_n(z)\}$ define the orbit $\{\widehat{z}(\lambda, z_0)\}$. Let $\lambda = \psi(x) : B(x_0, \rho) \subseteq \mathbb{R}^n \to int\Lambda \subseteq \mathbb{R}^n$ be such that $\psi(x_0) = 0$ and $\widehat{x}(\psi(x), z_0) = x$. Denote $p(x) = \widehat{p}(\psi(x), z_0)$, $u(x) = \widehat{u}(\psi(x), z_0)$ and using the equation

$$H_0(\widehat{z}(\lambda, z_0)) = H_0(z_0), \; \lambda \in \Lambda$$

we get

$$H_0(z(x)) = H_0(z_0), \; \forall\, x \in B(x_0, \rho) \subseteq \mathbb{R}^n$$

where $z(x) = (x, p(x), u(x))$. Using (2.103) written on corresponding components we find that $\partial_\lambda \widehat{u}(\lambda, z_0) q_i(\lambda) = \widehat{p}(\lambda, z_0)\partial_\lambda \widehat{x}(\lambda, z_0) q_i(\lambda))$, $i \in \{1, ..., n\}$ and

$$\partial_\lambda u(\lambda, z_0) = \widehat{p}(\lambda, z_0)\partial_\lambda \widehat{x}(\lambda, z_0), \; \lambda \in \Lambda \tag{2.108}$$

is satisfied, where $\widehat{p}(\lambda) \in \mathbb{R}^n$ is a row vector. On the other hand, a direct computation applied to $u(\widehat{x}(\lambda, z_0)) = \widehat{u}(\lambda, z_0)$ leads us to

$$\partial_x u(\widehat{x}(\lambda, z_0)).\partial_\lambda \widehat{x}(\lambda, z_0) = \partial_\lambda \widehat{u}(\lambda, z_0) \tag{2.109}$$

and using (2.105) we may and do multiply by the inverse matrix $[\partial_\lambda \widehat{x}(\lambda, z_0)]^{-1}$ in both equations (2.108) and (2.109). We get $\widehat{p}(\lambda, z_0) = \partial_x u(\widehat{x}(\lambda, z_0))$, for any $\lambda \in B(z_0, 2\rho)$ ($\rho > 0$ sufficiently small) which stands for

$$\partial_x u(x) = p(x),\ x \in B(z_0, \rho) \subseteq \mathbb{R}^n$$

provided $\lambda = \psi(x)$ is used. The conclusions (2.107) tell us that the gradient $\partial_x[H_0(x, p(x), u(x))]$ is vanishing and the proof is complete. □

Remark 2.3.8. *Taking $\{Z_1, ..., Z_m\} \subseteq K$ is involution we get the property* (2.103) *fulfilled (see §5 of ch II)*

$$0 = \begin{cases} \{H_i, H_0\}(\widehat{z}(z, z_0)) \\ < \partial_z H_0(\widehat{z}(\lambda, z_0)), Z_i(\widehat{z}(\lambda, z_0)) > \\ - < T(p)\partial_z H_0(\widehat{z}(\lambda, z_0)), \partial_z H_i(\widehat{z}(\lambda, z_0)) > \\ - < \partial_z H_{i,}(\widehat{z}(\lambda, z_0)), Z_0(\widehat{z}(\lambda, z_0)) >,\ i \in \{1, ..., m\} \end{cases} \tag{2.110}$$

It shows that $\{H_1, ..., H_m\} \subseteq \mathcal{H} = \mathcal{C}^\infty(\mathbb{R}^{2n+1}, \mathbb{R})$ defining $\{Z_1, ..., Z_m\} \subseteq K$ can be found as first integrals for a system of ODE

$$\frac{dz}{dt} = Z_0(z), Z_0(z) = T(p)\partial_z H_0(z) \in K$$

corresponding to $H_0(z)$.

2.3.4 The linear Case: $H_0(x, p) =< p, f_0(x) >$

With the same notations as in §4.3 we define $z = (x, p) \in \mathbb{R}^{2n}$ and $\mathcal{H} = \{H(z) =< p, f(x) >, p \in \mathbb{R}^{2n}, f \in \mathcal{C}^\infty(\mathbb{R}^{2n}, \mathbb{R}^{2n})\}$. The linear space of characteristic fields $K \subseteq \mathcal{C}^\infty(\mathbb{R}^{2n}, \mathbb{R}^{2n})$ is the image of a linear mapping $S : D\mathcal{H} \to K$, where

$$D\mathcal{H} = \{\partial_z H : H \in \mathcal{H}\},\ S(\partial_z H)(z) = T\partial_z H(z),\ z \in \mathbb{R}^{2n} \tag{2.111}$$

$$T = \begin{pmatrix} O & I_n \\ -I_n & O \end{pmatrix} for\, each\, Z \in K \tag{2.112}$$

is given by

$$Z(z) = \begin{pmatrix} \partial_p H(z) \\ -\partial_x H(z) \end{pmatrix} = \begin{pmatrix} f(x) \\ -\partial_x < p, f(x) > \end{pmatrix} \tag{2.113}$$

Let $z_0 = (p_0, x_0) \in \mathbb{R}^{2n}$ and $B(z_0, 2\rho) \subseteq \mathbb{R}^{2n}$ be fixed and consider the following linear equation of $p \in \mathbb{R}^n$

$$H_0(x, p) =< p, f_0(x) >= H_0(z_0) \text{for any } z \in D \subseteq B(z_0, 2\rho) \tag{2.114}$$

where $f_0 \in \mathcal{C}^1(\mathbb{R}^{2n}, \mathbb{R}^{2n})$. We are looking for $D \subseteq B(z_0, 2\rho) \subseteq \mathbb{R}^{2n}$ as an orbit.

$$\widehat{z}(\lambda, z_0) = G_1(t_1) \circ ... \circ G_m(t_m)(z_0),\ \lambda = \{t_1, ..., t_m\} \in \Lambda = \prod_1^m [-a_i, a_i] \tag{2.115}$$

where $G_i(\tau)(y)$, $y \in B(z_0,\rho)$, $\tau \in [-a_i,a_i]$, is the local flow generated by some $Z_i \in K$, $i \in \{1,...,m\}$, $m \leqslant n$. Assuming that

$$\{Z_1,...,z_m\} \subseteq K \text{ are in involution over } \mathcal{C}^\infty(B(z_0,2\rho),\mathbb{R}) \tag{2.116}$$

where

$$Z_i(z) = \begin{pmatrix} f_i(x) \\ -\partial_x < p, f_i(x) > \end{pmatrix} \text{and} \{f_1,...,f_m\} \subseteq \mathcal{C}^\infty(\mathbb{R}^{2n},\mathbb{R}^{2n})$$

are in involution over $\mathcal{C}^\infty(B(x_0),\mathbb{R})$ we define

$$D = \{z \in B(z_0,2\rho) : z = \widehat{z}(\lambda,z_0), \lambda \in \Lambda\} \subseteq R^{2n} \tag{2.117}$$

where the orbit $\{\widehat{z}(\lambda,z_0) : \lambda \in \Lambda\}$ is given in (2.115). Notice that the orbit (2.115) is represented by

$$\widehat{z}(\lambda,z_0) = (\widehat{x}(\lambda,z_0),\widehat{p}(\lambda,z_0)),\ \lambda \in \Lambda,\ z_0 = (x_0,p_0) \subseteq R^{2n} \tag{2.118}$$

where the orbit $\widehat{x}(\lambda,z_0), \lambda \in \Lambda, in\, \mathbb{R}^n$ verifies

$$\widehat{x}(\lambda,x_0) = F_1(t_1) \circ ... \circ F_m(t_m)(x_0),\ \lambda = \{t_1,...,t_m\} \in \Lambda \tag{2.119}$$

Here $F_i(\tau)(x)$, $x \in B(x_0,\rho) \subseteq R^n$, $\tau \in [-a_i,a_i]$, is the local flowgenerated by the vector field $f_i \in \mathcal{C}^\infty(\mathbb{R}^n,\mathbb{R}^n)$ and $\{f_1,...,f_m\} \subseteq \mathcal{C}^\infty(\mathbb{R}^n,\mathbb{R}^n)$ are in involution over $\mathcal{C}^\infty(B(x_0,\rho))$.

Remark 2.3.9. *Denote by $N_{x_0} \subseteq \mathbb{R}^n$ the set consisting of all points $\{\widehat{x}(\lambda,x_0) : \lambda \in \Lambda\}$ using the orbit defined in* (2.119)*. Assuming that$\{f_1(x_0),...,f_m(x_0)\} \subseteq R^n$ are linearly independent then $\{f(x_0),...,f_m(x)\} \subseteq R^n$ are linearly independent for any $x \in B(x_0,\rho)(\rho$ sufficiently small) and the subset $N_{x_0} \subseteq \mathbb{R}^n$ can be structured as an m-dimensional smooth manifold. In addition,the set $D \subseteq \mathbb{R}^{2n}$ defined in* (2.117) *can be verified as an image of smooth mapping $z(x) = (x,p(x)) : N_{x_0} \to D$, if $p(x) = \widehat{p}(\psi(x),z_0)$ and $\lambda = \psi(x) : B(x_0,\rho) \to \Lambda$ is unique solution of the algebraic equation $\widehat{x}(\lambda,x_0) = x \in B(x_0,\rho)$.*

Definition 2.3.10. *The orbit $\{\widehat{z}(\lambda,z_0) : \lambda \in \Lambda\}$ defined in* (2.115) *is a parameterized solution of the linear equation* (2.114) *if $H_0(\widehat{z}(\lambda,z_0)) = H(z_0)\,,\forall\,\lambda \in \Lambda$.*

Proposition 2.3.11. *Let $z_0 = (x_0,p_0) \subseteq R^{2n}$ and $B(z_0,2\rho) \subseteq R^{2n}$ be fixed such that the hypothesis* (2.116) *is fulfilled.For $\{Z_1,...,Z_m\} \subseteq K$ given in* (2.116) *assume in addition, that $\{f_1,...,f_m\}$ are commuting with $\{f_0\}$ i.e $[f_i,f_0](x) = 0$, $x \in B(x_0,2\rho)$, $i \in \{1,...,m\}$. Then the orbit $\{\widehat{z}(\lambda,z_0) : \lambda \in \Lambda\}$ defined in* (2.115) *is a parameterized solution of equation* (2.114)*, where $D \in \mathbb{R}^{2n}$ is given in* (2.117)*.*

Proof. By hypothesis ,the Lie algebra $L(Z_1,...,Z_m) \subseteq \mathcal{C}^\infty(\mathbb{R}^{2n},\mathbb{R}^{2n})$ is of finite type (locally) and $\{Z_1,...,Z_m\}$ is a system of generators in $L(Z_1,...,Z_m)$. As a consequence $L(Z_1,...,Z_m)$ is $(f.g.\circ;z_0)$ and using the algebraic representation of the gradient system associated with $\{Z_1,...,Z_m\}$ we get $\{q_1,...,q_m\} \subseteq \mathcal{C}^\infty(\Lambda;\mathbb{R}^n)$ such that

$$\partial_\lambda \widehat{z}(\lambda,z_0).q_i(\lambda) = Z_i(\widehat{z}(\lambda,z_0)),\ \lambda \in \Lambda,\ i \in \{1,...,m\},\ q_1(\lambda),...,q_m(\lambda) \in \mathbb{R}^m \tag{2.120}$$

are linearly independent for any $\lambda \in \Lambda$.
The meaning of $(f.g.\theta; z_0)$ Lie algebra and the conclusion (2.120) are explained in the next section. For the time being we use (2.120) and taking Lie derivatives of the scalar equation $H_0(\widehat{z}(\lambda, z_0)) = H_0(z_0)$, $\lambda \in \Lambda$, we get

$$0 = \partial_\lambda H(\widehat{z}(\lambda, z_0)).q_i(\lambda) =< \partial_z H(\widehat{z}(\lambda, z_0)), Z_i(\widehat{z}(\lambda, z_0)) >, \text{ for any } i \in \{1, ..., m\} \quad (2.121)$$

and $\lambda \in \Lambda$. By definition

$$\partial_z H_0 = \begin{pmatrix} \partial_x < p, f_0(x) > \\ f_0(x) \end{pmatrix}, \, and \, Z_i(z) = \begin{pmatrix} f_i(x) \\ -\partial_x < p, f_0(x) > \end{pmatrix}, \, i \in \{1, ..., m\}$$

which allows us to rewrite (2.121) as follows

$$< \widehat{p}(\lambda, z_0), [f_i, f_0](\widehat{x}(\lambda, x_0)) >= 0, \, i \in \{1, ..., m\}, \, \lambda \in \Lambda \quad (2.122)$$

Using $[f_i, f_0](x) = 0$, $x \in B(x_0, 2\rho)$, $i \in \{1, ..., m\}$ we obtain that (2.121) is fulfilled and it implies

$$\partial_\lambda H_0(\widehat{z}(\lambda, z_0)) = 0, \, \forall \, \lambda \in \Lambda \quad (2.123)$$

provided (2.120) is used. In conclusion, the scalar equation $H_0(z) = H_0(z_0)$ for any $z \in D \subseteq R^{2n}$, is satisfied, where D is defined in(2.117) and the proof is complete. □

Remark 2.3.12. *The involution condition of* $\{Z_1, ..., Z_m\} \subseteq K$ *is satisfied if*

$$\{f_1, ..., f_m\} \subseteq \mathcal{C}^\infty(B(x_0, 2\rho); \mathbb{R}^n)$$

are in involution over $\mathbb{R}$. *In this respect, using the particular form of*

$$Z_i(z) = \begin{pmatrix} f_i(x) \\ A_i(x)p \end{pmatrix}, A_i(x) = -[\partial_x f_i(x)]^* \, i \in \{1, ..., m\}$$

we compute a Lie bracket $[Z_i, Z_j]$ *as follows*

$$[Z_i, Z_j](z) = \begin{pmatrix} [f_i, f_j](x) \\ P_{ij}(z) \end{pmatrix}, \quad i, j \in \{1, ..., m\} \quad (2.124)$$

Here $[f_i, f_j]$ *is the Lie bracket from* $\mathcal{C}^\infty(\mathbb{R}^{2n}, \mathbb{R}^{2n})$ *and*

$$P_{ij}(z) \begin{cases} = [\partial_x(A_j(x)p)].f_i(x) + A_j(x)A_i(x)p - [\partial_x(A_i(x)p)]f_j(x) \\ = -A_i(x)A_j(x)p \\ = \partial_x[< A_j(x)p, f_i(x) > - < A_i(x)p, f_j(x) >] \\ = -\partial_x < p, [f_i, f_j](x) > \end{cases} \quad (2.125)$$

where

$$P_{ij}(z) = [\partial_z(A_j(x)p)]Z_i(z) - [\partial_z(A_i(x)p)]Z_j(x), \, z \in \mathbb{R}^{2n} \quad (2.126)$$

is used. Notice that (2.125) *allows one to write* (2.3.12) *as a vector field from* K

$$[Z_i, Z_j](z) = T\partial_z H_{ij}(z), \, z = (x, p) \in \mathbb{R}^{2n} \quad (2.127)$$

where $H_{ij} =< p, [f_i, f_j](x) >$, $i, j \in \{1, ..., m\}$. *As a consequence, assuming that* $\{f_1, ..., f_m\}$ *are in involution over* $\mathbb{R}$ *we get that* $L(f_1, ..., f_m)$ *and* $L(Z_1, ..., Z_m)$ *are finite dimensional with* $\{Z_1, ..., Z_m\}$ *in involution over* $\mathbb{R}$.

2.3.5 The Case $H_0(x, p, u) = H_0(x, p)$; Stationary Solutions

Denote $\mathcal{C}^\infty(\mathbb{R}^{2n}, \mathbb{R}^n)$, $z = (x, p) \in \mathbb{R}^{2n}$ and define the linear space of characteristic fields $K \subseteq \mathcal{C}^\infty(\mathbb{R}^{2n}, \mathbb{R}^n)$, $z = (x, p) \in \mathbb{R}^{2n}$ by

$$K = S(D\mathcal{H}),\ D\mathcal{H} = \{\partial_z H : H \in \mathcal{H}\} \tag{2.128}$$

Here the linear mapping : $D\mathcal{H} \to K$ is given by

$$S(\partial_z H)(z) = T\partial_z H(z),\ H \in \mathcal{H},\ z \in \mathbb{R}^{2n} \tag{2.129}$$

and T is the simplectic matrix

$$T = \begin{pmatrix} O & I_n \\ -I_n & O \end{pmatrix}, T^2 = \begin{pmatrix} -I_n & O \\ O & -I_n \end{pmatrix} \tag{2.130}$$

Let $z_0 = (x_0, p_0) \in \mathbb{R}^{2n}$ and $B(z_0, 2\rho) \subseteq \mathbb{R}^{2n}$ be fixed and consider the following nonlinear equation

$$H_0(z) = H_0(z_0),\ \forall\, z \in D \subseteq B(z_0, 2\rho) \tag{2.131}$$

where $H_0 \in \mathcal{C}^1(B(z_0, 2\rho); \mathbb{R})$ is given and $D \subseteq B(z_0, 2\rho)$ has to be found. A solution for (2.131) uses the following assumption

$$\text{there exist } Z_1, ..., Z_m \in K \text{ such that } \{Z_1(z), ..., z_m(z) : z \in B(z_0, 2\rho)\} \tag{2.132}$$

in involution over $\mathcal{C}^\infty(B(z_0, 2\rho); \mathbb{R})$. Assuming that (2.132) is fulfilled then a parameterized solution for (2.131) uses the following orbit

$$\widehat{z}(\lambda, z_0) = G_1(t_1) \circ ... \circ G_m(t_m)(z_0),\ \lambda = (t_1, ..., t_m) \in \Lambda = \prod_1^m [-a_i, a_i] \tag{2.133}$$

where

$$G_i(\tau)(y),\ y \in B(z_0, \rho),\ \tau \in [-a_i, a_i]$$

is the local flow generated by $Z_i \in K$ given in (2.132). By definition

$$Z_i(z) = T\partial_z H_i(z) = \begin{pmatrix} \partial_p H_i(x, p) \\ -\partial_x H_i(x, p) \end{pmatrix},\ i \in \{1, ..., m\} \tag{2.134}$$

and we recall that, in this case, the linear space of characteristic fields K is closed under the lie bracket. As a consequence, each Lie product $[Z_i, Z_j]$ can be computed by

$$[Z_i, Z_j](z) = T\partial_z H_{ij}(z),\ i, j \in \{1, ..., m\}, z \in \mathbb{R}^{2n} \tag{2.135}$$

where $H_{ij}(z) = \{H_i, H_j\}(z) =< \partial_z H_{ij}(z), Z_i(z) >= - < \partial_z H_i, Z_j(z) >$ stands for Poisson bracket associated with two $\mathcal{C}^1$ scalar functions. In addition, assuming (2.132) we get that the Lie algebra $L(Z_1, ..., Z_m) \subseteq \mathcal{C}^\infty(B(z_0, 2\rho), \mathbb{R}^n)$ is of the finite type (see in next section) allowing one to use the algebraic representation of the corresponding gradient system. We get that$\{q_1, ..., q_m\} \subseteq \mathcal{C}^\infty(\Lambda, \mathbb{R}^m)$ will exist such that

$$\partial_\lambda \widehat{z}(\lambda, z_0) q_i(\lambda) = Z_i(\widehat{z}(\lambda, z_0)),\ i \in \{1, ..., m\}\ , \lambda \in \Lambda, q_1(\lambda), ..., q_m(\lambda) \in \mathbb{R}^m \quad (2.136)$$

are linearly independent, $\lambda \in \Lambda$, where $\{\widehat{z}(\lambda, z_0) : \lambda \in \Lambda\}$ is the fixed orbit given in (2.133). Denote $D = \{z \in B(z_0, 2\rho) : z = \widehat{z}(\lambda, z_0), \lambda \in \Lambda\}$ and the orbit given in (2.133) is a parameterized solution of the equation (2.131) iff

$$\partial_\lambda [H_0(\widehat{z}(\lambda, z_0))] q_i(\lambda) = 0,\ i \in \{1, ..., m\} \quad (2.137)$$

Using (2.136) we rewrite (2.137)as follows

$$0 =< \partial_\lambda H_0(\widehat{z}(\lambda, z_0)), Z_i(\widehat{z}(\lambda, z_0)) >= \{H_i, H_0\}(\widehat{z}(\lambda, z_0)),\ \lambda \in \Lambda, \forall\ i \in \{1, ..., m\} \quad (2.138)$$

It is easily seen that (2.138) is fulfilled provided

$$H_1, ..., H_m \in \mathcal{C}^\infty(B(z_0, 2\rho), \mathbb{R}^n)$$

can be found such that

$$0 = \{H_i, H_0\}(z) =< \partial_z H_0(z), T\partial_z H_i(z) >=< Z_0(z), \partial_z H_i(z) > \quad (2.139)$$

for each $i \in \{1, ..., m\}$ $z \in B(z_0, 2\rho)$ and $Z_i(z) = T\partial_z H_i(z)\ , i \in \{1, ..., m\}$ are in involution for $z \in B(z_0, 2\rho)$. In particular ,the equations in (2.139) tell us that $\{H_1, ..., H_m\}$ are first integrals for the vector field $Z_0(z) = T\partial_z H_0(z)$.

Remark 2.3.13. *Taking $H_0 \in \mathcal{C}^2(B(z_0, 2\rho; \mathbb{R}))$(instead of $H_0 \in \mathcal{C}^1$) we are in position to rewrite the equation* (2.139) *using Lie product $[Z_0, Z_j]$ (see* (2.135)*) as follows*

$$0 = \begin{cases} [Z_0, Z_j](z),\ \forall\, j \in \{1, ..., m\},\ z \in B(z_0, 2\rho) \\ \{H_i, H_0\}(z_0) =< Z_0(z_0), \partial_z H_i(z_0) >,\ i \in \{1, ..., m\} \end{cases} \quad (2.140)$$

We conclude these considerations.

Proposition 2.3.14. *Let $z_0 = (x_0, p_0) \in \mathbb{R}^{2n}$ and $B(z_0, 2\rho) \subseteq \mathbb{R}^{2n}$ be fixed such that the hypothesis* (2.151) *and* (2.138) *are fulfilled. Then $\{\widehat{z}(\lambda, z_o) : \lambda \in \Lambda\}$ defined in* (2.133)*is a parameterized stationary solution of the nonlinear equation* (2.131).

Remark 2.3.15. *A standard solution for the nonlinear equation*

$$H_0(x, p(x)) = const,\ \forall\, x \in D \subseteq \mathbb{R}^n$$

can be found from a parameterized solution provided the hypothesis (2.132) *is stated*

with $m = n$ and assuming, in addition, that $X_1(z_0), ..., X_n(z_0) \in \mathbb{R}^n$ from

$$Z_i(z) = \begin{pmatrix} X_i(z) \\ P_i(z) \end{pmatrix}, \, i \in \{1, ..., n\}$$

are linearly independent. It leads us to the equations

$$H_0(\widehat{x}(\lambda, z_0), \widehat{p}(\lambda, z_0)) = 0 \;\forall \lambda \in \Lambda = \prod_1^n [-a_i, a_i] \tag{2.141}$$

where the orbit $\widehat{z}(\lambda, z_0) = (\widehat{x}(\lambda, z_0), \widehat{p}(\lambda, z_0))$ satisfies the implicit functions theorem when the algebraic equations

$$\widehat{x}(\lambda, z_0) = x \in B(x_0, \rho) \subseteq \mathbb{R}^n$$

are involved,Find a smooth mapping

$$\lambda = \psi(x) : B(z_0, 2\rho) \to \lambda$$

such that

$$\widehat{x}(\psi(x), z_0) = x$$

and define

$$p(x) = \widehat{p}(\psi(x), z_0), \, x \in B(x_0, \rho) \subseteq \mathbb{R}^n$$

Then (2.141) *written for $\lambda = \psi(x)$ becomes*

$$H_0(x, p(x)) = const = H_0(x_0, p_0), \,\forall x \in B(x_0, \rho) \tag{2.142}$$

and $p(x) : B(x_0, \rho) \to \mathbb{R}^n$ is the standard solution. Assuming that (2.142)*is established it implies that $\{p(x) : x \in B(x_0, \rho)\}$ is a smooth solution of the following system of the first order causilinear equations*

$$\partial_x H_0(x, p(x)) + [\partial_x p^*(x)]^* . \partial_p H_0(x, p(x)) = 0, \, x \in B(x_0, \rho) \tag{2.143}$$

whose solution is difficult to be obtained using characteristic system method. In addition,if $[\partial_x p(x)]$ is a symmetric matrix $(p(x) = \partial_x u(x), u \in \mathcal{C}^2(D; \mathbb{R}))$ then the first order system (2.143) *becomes*

$$\begin{pmatrix} \partial_{x_1} H_0(x, p(x)) \\ . \\ . \\ . \\ \partial_{x_n} H_0(x, p(x)) \end{pmatrix} + \begin{pmatrix} < \partial_x p_1(x) \partial_p H_0(x, p(x)) > \\ . \\ . \\ . \\ < \partial_x p_n(x) \partial_p H_0(x, p(x)) > \end{pmatrix} = 0 \tag{2.144}$$

for any $x \in B(x_0, \rho) \subseteq \mathbb{R}^n$, where $p(x) = (p_1(x), ..., p_n(x))$. In this case, the smooth mapping $y = p(x)$, $x \in B(x_0, \rho)$, can be viewed as a coordinate transformation in $\mathbb{R}^n$ such that any local solution $(\widehat{x}(t, \lambda), \widehat{p}(t, \lambda))$ of the Hamilton system

$$\begin{cases} \frac{dx}{dt} = & \partial_p H_0(x, p) \;\; x(0) = \lambda \in B(x_0, a) \subseteq B(x_0, \rho) \\ \frac{dp}{dt} = & -\partial_x H_0(x, p) \;\; p(0) = p(\lambda); t \in (-\alpha, \alpha) = I_\alpha \end{cases} \tag{2.145}$$

has the property $\widehat{p}(t,\lambda) = p(\widehat{x}(t,\lambda))$, $(t,\lambda) \in I_\alpha \times B(x_0,a)$.

2.3.6 Some Problems

Problem 1. For the linear equation $H_0(x,p) =< p, f_0(x) >= c$ define an extended parameterized solution

$$\widehat{Z}^e(\lambda,z_0) = (\widehat{x}(\lambda,z_0),\widehat{p}(\lambda,z_0),\widehat{u}(\lambda,z_0)) \in \mathbb{R}^{2n+1},\ \lambda \in \Lambda = \prod_1^m [-a_i,a_i]$$

such that

$$\partial_\lambda \widehat{u}(\lambda,z_0) = [\widehat{p}(\lambda,z_0)]^* \partial_\lambda \widehat{x}(\lambda,z_0) \tag{2.146}$$

where $z_0 = (x_0,p_0,u_0)$.
Hint. Define the extended vector fields

$$Z_i^e \in \mathcal{C}^\infty(\mathbb{R}^{2n+1},\mathbb{R}^{2n+1}) Z_i^e(z^e) = \begin{pmatrix} Z_i(x,p) \\ H_i(x,p) \end{pmatrix}, z_e = (x,p,u) = (z,u)$$

where $H_i(x,p) =< p, f_i(x) >$ and $Z_i(x,p) = T\partial_z H_i(x,p)$, $z = (x,p)$, $i \in \{1,...,m\}$, where the matrix T is defined in (2.112). Assume the hypothesis (2.116) of section 2.4.4 fulfilled and define the orbit

$$\widehat{Z}^e(\lambda,z_0) = G_1^e(t_1) \circ ... \circ G_m^e(t_m)(z_0),\ \lambda = (t_1,...,t_m) \in \Lambda = \prod_1^m [-a_i,a_i] \tag{2.147}$$

where $G_i^e(\tau)(y^e)$ is the flow generated by Z_i^e. Prove that under the hypothesis assumed in proposition (2.3.11) we get $\{\widehat{z}^e(\lambda,z_0) : \lambda \in \Lambda\}$ as a parameterized solution of the linear equation $H_0(x,p) = const$ satisfied the conclusion (2.146)
Problem 2. For the nonlinear equation $H_0(x,p) =$ const define an extended parameterized solution

$$\widehat{z}^e(\lambda,z_0^e) = (\widehat{x}(\lambda,z_0^e),\widehat{p}(\lambda,z_0^e)),\widehat{u}(\lambda,z_0^e) \in \mathbb{R}^{2n+1}, \lambda \in \Lambda = \prod_1^m [-a_i,a_i]$$

such that

$$\partial_\lambda \widehat{u}(\lambda,z_0) = [\widehat{p}(\lambda,z_0)]^* \partial_\lambda \widehat{x}(\lambda,z_0),\ \lambda \in \Lambda \,\text{where}\, Z_0^e = (x_0,p_0,u_0) \in \mathbb{R}^{2n+1} \tag{2.148}$$

is fixed.
Hint

$$Z_i^e(z^e) = \begin{pmatrix} Z_i(z) \\ < p, X_i(z) > \end{pmatrix} \mathit{where}\, Z_i(z) \begin{pmatrix} X_i(z) \\ P_i(z) \end{pmatrix} \tag{2.149}$$

is obtained from a smooth $H_i \in \mathcal{C}^\infty(\mathbb{R}^{2n}, \mathbb{R})$ as follows

$$Z_i(z) = T\partial_z H_i(z),\, i \in \{1, ..., m\}$$

given in (2.130) defined the extended orbit

$$\widehat{z}^e(\lambda, z_0^e) = G_1^e(t_1) \circ ... \circ G_m^e(t_m)(z_0^e),\, \lambda = (t_1, ..., t_m) \in \Lambda = \prod_1^m [-a_i, a_i] \qquad (2.150)$$

where $G_i^e(\tau)(y^e)$ is the local flow $\{Z_i^e\}$. Assume that

$$\{Z_1^e(z^e), ..., Z_m^e(z^e) : z^e \in B(z_0, 2\rho) \subseteq \mathbb{R}^{2n+1}\} \qquad (2.151)$$

are in involution over $\mathcal{C}^\infty(B(z_0, 2\rho); \mathbb{R})$ and then prove the corresponding proposition (2.3.14) of assuming that replaces the condition (2.132).

2.4 Overdetermined system of First Order PDE and Lie Algebras of Vector Fields

As one may expect, a system of first order partial differential equations

$$< \partial_x S(x), g_i(x) >= 0,\, i \in \{1, ..., m\}\, g_i \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^n) \qquad (2.152)$$

has a nontrivial solution $S \in \mathcal{C}^2(B(z_0, \rho) \subseteq \mathbb{R}^{2n})$ provided S is nonconstant on the ball $B(z_0, \rho) \subseteq \mathbb{R}^{2n+1}$ and

$$\dim L(g_1, ..., g_m(x_0)) = k < n \qquad (2.153)$$

Here $L(g_1, ..., g_m) \subseteq \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^n)$ is the Lie algebra determined by the given vector fields $\{g_1, ..., g_m\} \subseteq \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^n)$, it can be viewed as the smallest Lie algebra $\Lambda \subseteq \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^n)$ containing $\{g_1, ..., g_m\}$.

Definition 2.4.1. *A real algebra $\Lambda \subseteq \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^n)$ is Lie algebra if the multiplication operation among vector fields $X, Y \subseteq \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^n)$ is given by the corresponding Lie bracket $[X, Y](x) = \frac{\partial Y}{\partial x}(x)X(x) - \frac{\partial X}{\partial x}(x)Y(x)$, $x \in \mathbb{R}^n$. By a direct computation we may convince ourselves that the following two properties are valid*

$$[X, Y] = -[Y, X],\, \forall\, X, Y \in \Lambda$$

and

$$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0,\, \forall\, X, Y, Z \in \Lambda$$

(Jacobi s' identity)

Remark 2.4.2. *The system* (2.152) *is overdetermined when $m > 1$. Usually a nontrivial solution $\{S(x), x \in B(x_0, \rho)\}$ satisfies the system* (2.152) *only on a subset*

$M_{x_0} \subseteq B(x_0,\rho) \subseteq \mathbb{R}^n$, which can be structured as a smooth manifold satisfying $dimM_{x_0} = k$ (see (2.153)); it will be deduced from an orbit of local flows.

Definition 2.4.3. *Let $\Lambda \subseteq \mathcal{C}^\infty(\mathbb{R}^n,\mathbb{R}^n)$ be a Lie algebra and $x_0 \in \mathbb{R}^n$ fixed. By an orbit of the origin x_0 of Λ we mean a mapping (finite composition of flows)*

$$G(p,x_0) = G_1(t_1)\circ....\circ G(t_k)(x_0)\,, p = (t_1,t_k) \in D_k = \prod_1^k[-a_i,a_i] \qquad (2.154)$$

where $G_i(t)(x), t \in (-a_i,a_i), x \in V(x_0)$, is the local flow generated by some $g_i \in \Lambda, i \in \{1...k\}$.

Definition 2.4.4. *We say that $\Lambda \subseteq \mathcal{C}^\infty(\mathbb{R}^n,\mathbb{R}^n)$is finitely generated with respect to the orbits of the origin $x_0 \in \mathbb{R}^n, (f,g,0,x_0)$ if $\{Y_i,...,Y_m\} \subseteq \Lambda$ will exist such that any $Y \in \Lambda$ along on arbitrary orbit $G(p,x_0), p \in D_k$ can be written*

$$Y(G(p,x_0)) = \Sigma_{j=1}^M a_j(p)Y_j(G(p,x_0)) \qquad (2.155)$$

with $a_j \in \mathcal{C}^\infty(D_k,\mathbb{R})$depending on Y and $G(p,x_0), p \in D_k; \{Y_I,...Y_M\} \subseteq \Lambda$ will be called a system of generators.

Remark 2.4.5. *It is easily seen that $\{g_1,..g_m\} \subseteq \mathcal{C}^\infty(\mathbb{R}^n),\mathbb{R}^n)$ in involution determine a Lie algebra $L(g_I,..g_m)$ which is $(f.g.o.x_0)$ for any $x_o \in \mathbb{R}^n$, where the involution properly means $[g_i,...,g_j](x) = \sum_{k=1}^m a_{ij}^k(x)g_k(x)\,\forall, i,j \in \{1,...m\}$ for some, $a_{ij}^k in\mathcal{C}^\infty(\mathbb{R}^n), \{g_1,...,g_m\}$ will be a system of generators. A nontrivial solution of the system (1) will be constructed assuming that*

$$L(g_1,...g_m)\,is\,a\,(f,g,o;x_o)\,Lie\ algebra \qquad (2.156)$$

$$dimL(g_1,...g_m)(x_0) = k < n \qquad (2.157)$$

In addition, the domain $V(x_0) \subseteq \mathbb{R}^n$ on which a non trivial solution satisfies(2.152) will be defined as an orbit starting the origin $x_o \in \mathbb{R}^n$

$$y(p) = G_1(t_1\circ...\circ G_M(t_M)(x_0),\, p = (t_1,...,t_M) \in D_M = \prod_1^M(-a_i,a_i) \qquad (2.158)$$

where $G_i(t)(x), t \in (-a_i,a_i)$, $x \in B(x_0,\rho) \subseteq \mathbb{R}^n$ is the local flow generated by $Y_i \in L(g_1,...,g_m)$, $i \in \{1,...,M\}$.Here $\{Y_1,...,Y_M\} \subseteq L(g_1,...,g_m)$ is fixed system of generators such that

$$\{Y_1(x_0),...,Y_k(x_0)\} \subseteq \mathbb{R}^n \qquad (2.159)$$

are linearly independent and $Y_j(x_0) = 0$, $j \in \{1,...,M\}$

2.4.1 Solution for Linear Homogeneous Over Determined System.

Theorem 2.4.6. *Assume that* $g_1, ...g_m \subseteq \mathcal{C}^\infty(\mathbb{R}^n), \mathbb{R}^n)$ *are given such that the hypotheses* (2.156) *and* (2.157) *are satisfied. Let* $\{Y_1, .., Y_M\} \subseteq L(g_1, ..g_m)$ *be a system of generators fulfilling*(2.159) *and define*

$$V(x_o) = \{x \in B(x_0, \rho) \subseteq \mathbb{R}^n : x = y(p), p \in D_M\} \subseteq \mathbb{R}^n \tag{2.160}$$

where $\{y(p) :\in D_m\}$ *is the orbit define in*(2.158). *Then there exists a smooth nontrivial function*
$S(x) : B(x_o, \rho) \to \mathbb{R}$ *such that the system* (2.152) *is satisfied for any* $x \in V(x_0)$*given in* (2.160)*,i.e*

$$< \partial_x S(x), g_i(x) >= 0 \ \forall x \in V(x_0) \ i \in \{1, ...m\} \tag{2.161}$$

Proof. To prove the conclusion (2.161) we need to show the gradient system associated with the orbit $\{y(p) : p \in D_M\}$ defined in (2.158) has a nonsingular algebraic representation

$$\begin{aligned} \frac{\partial y(p)}{\partial p} &= \{Y_1(.), X_2(t_1, .), ..., X_M(t_1, ..., t_{M-i}; .)\}(y(p)) \\ &= \{Y_1(.), Y_2(.), ..., Y_M\}(y(p))A(p)p \in D_M \end{aligned} \tag{2.162}$$

where the smooth matrix $A(p) = (a_{ij}(p))_{i,j\in\{1,M\}}$ satisfies

$$A(0) = I_M, a_{ij} \in \mathcal{C}^\infty(D_M) \tag{2.163}$$

(seem next theorem). On the other hand, we notice that according to the fixed system of generators $\{Y_1, ..., Y_M\} \subset L(g_1, ..., g_m)$ (see (2.159)), we may and rewrite the orbit(2.158) as follows

$$\{y(p) : p \in D_M\} = \{y(\widehat{p}) = G_1(t_1), ..., G_k(t_k)(x_0) : \widehat{p} = (t_1, ..., t_k) \in D_k\} \tag{2.164}$$

Using a standard procedure we redefine

$$M_{x_0} = \{y(p) : p \in B(0, \alpha)\} \subseteq B(x_0, \varrho) \subseteq \mathbb{R}^n, \text{ where } B(0, \rho) \subseteq D_k\} \tag{2.165}$$

as a smooth manifold satisfying $\dim M_{x_0} = k < n$ for which there exist $n - k$ smooth functions $\varphi_j \in \mathcal{C}^\infty(B(x_0, \varrho))$ fulfilling

$$\begin{cases} \varphi_j(x) = 0, \ j \in \{k+1, ..., n\}, x \in M_{x_0} \\ \{\partial_x \varphi_{k+1}(x_0), ..., \partial_x \varphi_n(x_0)\} \subseteq \mathbb{R}^n \text{ are linearly independent} \end{cases} \tag{2.166}$$

Using (2.163) and $\det A(p) \neq 0$ for p in a ball $B(0, \alpha) \subseteq D_M$ we find smooth $q_j \in$

$C^\infty(B(0,\alpha);\mathbb{R}^M), j \in \{1,..,M\}$, such that

$$\begin{cases} A(p)q_j(p) = e_j,\ p \in B(0,\alpha),\ \{l_1,...,l_M\} \subseteq \mathbb{R}^M \text{ is the canonical basis} \\ \frac{\partial y(p)}{\partial p} q_j(p) = Y_j(y(p)),\ p \in B(0,\alpha),\ j \in \{1,...,M\} \end{cases} \tag{2.167}$$

Using (2.166) and (2.167) we get $\varphi_j(y(p)) = 0$, $p \in B(0,\alpha) \subseteq D_M$, $j \in \{k+1,...,n\}$ and taking the Lie derivatives in the directions $q_1(p),...,q_M(p)$ we obtain

$$< \partial_x \varphi_j(y(\widehat{p})),\ Y_i(y(\widehat{p})) >= 0\, \widehat{p} \in B(0,\alpha) \subseteq D_k,\ i \in \{1,...,M\}\, \forall\, j \in \{k+1,...,n\} \tag{2.168}$$

By hypothesis, $L(g_1,...,g_m)$ is a $(f \cdot g \cdot o; x_0)$ Lie Algebra and each $g \in L\{g_1,...,g_m\}$ can be written

$$g(y(\widehat{p})) = \sum_{i=1}^{M} \alpha_i(\widehat{p}) Y_i(y(\widehat{p})),\ \widehat{p} \in B(0,\alpha) \subseteq \mathbb{R}^k,\ \alpha_i \in \mathcal{C}^\infty(B(0,\alpha)) \tag{2.169}$$

Using (2.168) and (2.169) we get

$$\begin{aligned} < \partial_x \varphi_j(x),\ g_i(x) > \quad &= \quad 0\, \forall\, x \in M_{x_0} \subseteq B(0,\varrho) \text{ for each} \\ & \qquad i \in \{1,...,m\} \text{ and } j \in \{k+1,...,n\} \end{aligned} \tag{2.170}$$

Here $\{\varphi_{k+1}(x),...,\varphi_n(x) : x \in B(0,\varrho)\}$ are $(n-k)$ smooth solutions satisfying the overdetermined system (2.152) along the k-dimensional manifold M_{x_0} in (2.165). The proof is complete. □

Remark 2.4.7. *The nonsingular algebraic representation of a gradient system*

$$\frac{\partial y}{\partial t_1} = Y_1(y),\ \frac{\partial y}{\partial t_2} = X_2(t_1;y), ..., \frac{\partial y}{\partial t_M} = X_M(t_1,...,t_{M-1};y) \tag{2.171}$$

associated with the system of generators $\{Y_1,...,Y_M\} \subseteq L(g_1,...,g_M)$ *relies on the assumption that* $L(g_1,...,g_M)$ *is a* $(f \cdot g \cdot o; x_0)$ *Lie algebra and the orbit* (2.158) *is a solution of* (2.171) *satisfying* $y(0) = x_0$. *The algorithm of defining a gradient system for which a given orbit is its solution was initiated in Chapter I of these lectures. Here we shall use the following formal power series*

$$\begin{cases} X_2(t_1;y) = (expt_1 adY_1)(Y_2)(y) \\ X_3(t_1,t_2;y) = (expt_1 adY_1) \cdot (expt_2 adY_2)(Y_3)(y) \\ \cdot \\ \cdot \\ \cdot \\ X_M(t_1,t_2,...,t_{M-1};y) = (expt_1 adY_1) \cdot ... \cdot (expt_{M-1} adY_{M-1})(Y_M)(y) \end{cases} \tag{2.172}$$

where the linear mapping $adY : L(g_1,...,g_M) \to L(g_1,...,g_M)$ *is defined by*

$$(adY)(Z)(y) = \frac{\partial Z}{\partial y}(y) \cdot Y(y) - \frac{\partial Y}{\partial y} Z(y), y \in \mathbb{R}^n\ for\, each\, Y, Z \in L(g_1,...,g_M) \tag{2.173}$$

Actually, the vector fields in the left hand side of (2.172) *are well defined by the*

following mappings

$$\begin{cases} X_2(t_1;y) = H_1(-t_1;y_1)(Y_2)(G_1(-t_1;y_1)), y_1 = y, \\ X_3(t_1,t_2;y) = H_1(-t_1;y_1)\cdot H_2(-t_2;y_2)(Y_3)(G_2(-t_2;y_2)) \\ . \\ . \\ . \\ X_M(t_1,t_2,...,t_{M-1};y) = H_1(-t_1;y_1)\cdot H_2(-t_2;y_2)\cdot ... \cdot H_{M-1}(-t_{M-1};y_{M-1})(Y_M)(y_M) \end{cases} \tag{2.174}$$

where

$$G_i(-t_i;y_i),\ t\in(-a_i,a_i),\ x\in V(x_0)$$

is the local flow generated by

$$Y_i, H_i(t,y) = [\frac{\partial G_i}{\partial y}(t;y)]^{-1}$$

and

$$y_{i+1} = G_i(-t_i;y_i), \in \{1,...,M-1\},\ \textit{where } y_1 = y.$$

The formal writing (2.172) *is motivated by the explicit computation we can perform using exponential formal series and noticing that the Taylor series associated with* (2.172) *and* (2.174) *coincide.*

2.5 Nonsingular Algebraic Representation of a Gradient System

Theorem 2.5.1. *Assume that $L(g_1,...,g_m)\subseteq \mathcal{C}^\infty(\mathbb{R}^n;\mathbb{R}^n)$ is a $(f\cdot g\cdot o;x_0)$ Lie algebra and consider $\{Y_1,...,Y_M\}\subseteq L(g_1,...,g_M)$ as a fixed system of generators. Define the orbit $\{y_p : p\in D_M\}$ as in* (2.158) *and associate the gradient system* (2.171) *where $\{X_2(t_1;y),...,X_M(t_1,t_2,...,t_{M-1};y)\}$ are defined in* (2.174)*. Then there exist an $(M\times M)$ nonsingular matrix $A(p) = (a_{ij}(p))_{i,j\in\{1,...,M\}}, a_{ij}\in\mathcal{C}^\infty(B(0,\alpha)\subseteq\mathbb{R}^n)$ such that*

$$\{Y_1(\cdot)X_2(t_1;\cdot),...,X_M(t_1,...,t_{M-1};\cdot)\}(y(p)) = \{Y_1(\cdot),Y_2(\cdot),...,Y_M(\cdot)\}(y(p))A(p) \tag{2.175}$$

for any $p\in B(0,\alpha)\subseteq\mathbb{R}^n$, and $A(0)=I_M$

Proof. Using the $(f\cdot g\cdot o;x_0)$ property of Lie algebra $L(g_1,...,g_M$ we fix the following $(M\times M)$ smooth matrices $B_i(p)$, $p\in D_M$, such that

$$adY_i\{Y_1,...,Y_M\}(y(p)) = \{Y_1,...,Y_M\}(y(p))\cdot B_i(p),\ i\in\{1,...,M\} \tag{2.176}$$

where $\{Y_1,...,Y_M\}\subseteq L(g_1,...,g_m)$ is a fixed system of generators. Let $\{Z_1(t_1,...,t_M)$:

$t \in [0, t_1]\}$ be the matrix solution

$$\frac{dZ_1}{dt} = Z_1 B_1(t_1 - t, t_2, ..., t_M), Z_1(0) = I_M \text{ (identity)} \tag{2.177}$$

The standard Picard's method of approximations applied in (2.177) allows one to write the solution $\{z, (t_1, ..., t_M) : t \in [0, t_1]\}$ as a convergent Volterra series, and by a direct computation we obtain

$$X_2(t_1; y(p)) = \{Y_1, ..., Y_M\}(y(p)) z_1(t_1, t_2, ..., t_M) e_2 \tag{2.178}$$

where $X_2(t_1; y)$ is defined in (2.174) and $\{e_1, ..., e_M\} \subset \mathbb{R}^n$ is the canonical basis. More precisely, denote

$$N_1(t; y) = H_1(-t; y)\{Y_1, ..., Y_M\}(G_1(-t; y)) \text{and} \, X_2(t_1; y) \tag{2.179}$$

can be written

$$X_2(t_1; y) = N_1(t_1; y) e_2 \tag{2.180}$$

Then we noticed that $N_1(t; y(p))$, $t \in [0, t_1]$ satisfies

$$\frac{dN_1}{dt} = N_1 B_1(t_1 - t, t_2, ..., t_M), N_1(0) = \{Y_1, ..., Y_M\}(y(p)) \tag{2.181}$$

where the matrix $B_1(p)$ is fixed in (2.121) and satisfies

$$adY_i\{Y_1, ..., Y_M\}(G_1(-t; y)) = \{Y_1, ..., Y_M\}(G_1(-t; y)) \cdot B_1(t_1 - t, t_2, ..., t_M) \tag{2.182}$$

for $y = y(p)$. The same method of Approximation shows that a convergent sequence $\{N_1^k(t) : t \in [0, t_1]\}_{k \geqslant 0}$ can be constructed such that

$$\begin{cases} N_1^0(t) = \{Y_1, ..., Y_M\}(y(p)) = N_1(0) \\ N_1^{k+1}(t) = N_1(0) +, \int_0^t N_1^k(s) B_1(t_1 - s, t_2, ..., t_M) ds, k = 0, 1, 2, ... \end{cases} \tag{2.183}$$

It is easily seen that

$$N_1^{k+1}(t) = N_1(o) Z_1^{k+1}(t; t_1, ..., t_M),\; t \in [0, t_1] \tag{2.184}$$

where $\{Z_1^k(t; t_1, ..., t_M) : t \in [0, t_1]\}_{k \geqslant 0}$ defines a solution in (2.177) and

$$\lim_{k \to \infty} Z_1^k(t; t_1, ..., t_M) = Z_1(t; t_1, ..., t_M) \tag{2.185}$$

uniformly in$t \in [0, t_1]$. Therefore using (2.184) and (2.185) we get that

$$N_1(t) = \lim_{k \to \infty} N_1^k(t) \;, t \in [0, t_1] \tag{2.186}$$

is the solution in (2.181) fulfilling

$$N_1(t; y) = \{Y_1, ..., Y_M\}(y) Z_1(t; t_1, ..., t_M),\; t \in [0, t_1] \tag{2.187}$$

if $y = y(p)$, and (2.178) holds. The next vector field $X_3(t_1, t_2; y)$ in (2.119) can be represented similarly and rewriting it as

$$X_3(t_1, t_2; y) = H_1(-t_1; y)\widehat{X_3}(t_2; y_2), \text{ for } y_1 = y(p) = y \tag{2.188}$$

$$y_2 = G_1(-t_1; y_1) = G_2(t_2) \circ ... \circ G_M(t_M)$$

and

$$\widehat{X_3}(t_2; y_2) = H_2(-t_2; y_2)Y_3(G_2(-t_2, y_2)) \tag{2.189}$$

can be represented using the same algorithm as in (2.178) and we get

$$\widehat{X_3}(t_2; y_2) = \{Y_1, ..., Y_M\}(y_2)Z_2(t_2; t_2, ..., t_M)e_3 \tag{2.190}$$

Here the nonsingular and smooth $M \times M$ $\{Z_2(t; t_2, ..., t_M : t \in [0, t_2]\}$ is the unique solution of the following linear matrix system

$$\frac{dZ_2}{dt} = Z_2B_2(t_2 - t, t_3, ..., t_M),\ Z_2(0) = I_M \tag{2.191}$$

where $B_2(t_2, ..., t_M)\, p \in D_M$ is fixed such that

$$adY_2\{Y_1, ..., Y_M\}(y_2) = \{Y_1, ..., Y_M\}(y_2)B_2(t_2, ..., t_M),\ p \in D_M \tag{2.192}$$

Denote

$$N_2(t, y_2) = H_{(-t, y_2)}\{Y_1, ..., Y_M\}G_2(-t; y_2),\ t \in [0, t_2] \tag{2.193}$$

and we obtain

$$\widehat{x}_3(t_2; y_2) = N_2(t_2; y_2)e_3 = \{Y_1, ..., Y_M\}(y_2)Z_2(t_2; t_2, ..., t_M)e_3 \tag{2.194}$$

That is to say, $\{N_2(t; y_2) : t \in [0, t_2]\}$ as a solution of

$$\frac{dN_2}{dt} = N_2B_2(t_2 - t, t_3, ..., t_M),\ N_2(0) = \{Y_1, ..., Y_M\}(y_2) \tag{2.195}$$

can be represented using a standard iterative procedure and we get

$$N_2(t; y_2) = \{Y_1, ..., Y_M\}(y_2)Z_2(t; t_2, ..., t_M)\ , t \in [0, t_2] \tag{2.196}$$

where the matrix Z_2 is the solution in(2.191). Now we use (2.194) into (2.188) and taking into account that $y_2 = G_1(-t_1, y_1)$ we rewrite

$$H_1(-t_1; y_1)\{Y_1, ..., Y_M\}(G_1(-t_1, y_1)) = N_1(t_1; y_1) \text{ for } y_1 = y(p) \tag{2.197}$$

where $\{N_1(t; y(p)) : t \in [0, t_1]\}$ fulfills(2.187). Therefore the vector field $X_3(t_1, t_2; y)$ in (2.174) satisfies

$$X_3(t_1, t_2; y) = \{Y_1, ..., Y_M\}(y)Z(t_1, t_1, ..., t_M) \times Z_2(t_2; t_2, ..., t_M)e_3 \tag{2.198}$$

for $y = y(p)$. An induction argument will complete the proof and each $X_j(p_j; y)$ in

(2.174) gets the corresponding representation

$$X_{j+1}(p_{j+1};y) = \{Y_1,...,Y_M\}(y)Z(t_1,t_1,...,t_M) \times ... \times Z_j(t_j;t_j,...,t_M)e_{j+1} \quad (2.199)$$

if $y = y(p)$ and $\{Z_j(t,t_j,...,t_M : t \in [0,t_j])\}$ $j \in \{2,...,M-1\}$ is the solution of the linear matrix equation

$$\frac{dZ_j}{dt} = Z_j B_j(t_j - t, t_{j+1},...,t_M),\ Z_j(0) = I_M \quad (2.200)$$

Here the smooth matrix $B_j(t_j,....,t_M)$, $p \in D_M$, is fixed such that

$$adY_j\{Y_1,...,Y_M\}(y_j) = \{Y_1,...,Y_M\}(y_j)B_j(t_j,...,t_M) \quad (2.201)$$

where $y_j = G_j(t_j) \circ ... \circ G_M(t_M)(x_0)$, $j = 1,...,M$. Now, for simplicity, denote the $M \times M$ smooth matrix

$$Z_j(p) = Z_1(t_1;t_1,...,t_M) \times ... \times Z_j(t_j;t_j,...,t_M),\ p = (t_1,...,t_M) \in D_M \quad (2.202)$$

$$j \in \{1,...,M-1\}\ and\ X_{j+1}(p_{j+1};y)$$

in (2.199) is written as

$$X_{j+1}(p_{j+1};y) = \{Y_1,...,Y_M\}(y)Z_j(p)e_{j+1}\ , if\ y = y(p),\ p \in\ D_M \quad (2.203)$$

In addition, define the smooth $(M \times M)$

$$A(p) = (e_1, Z_1(p)e_2,...,Z_{M-1}(p)e_M),\ p \in D_M \quad (2.204)$$

which fulfills $A(0) = I_M$ and represent the gradient system (2.171) as follows

$$\{Y_1(.), X_2(t_1,.),...,X_M(t_1,...,t_{M-1},.)\}(y) = \{Y_1,...,Y_M\}(y)A(p) \quad (2.205)$$

if $y = y(p)$, and the smooth $(M \times M)$ matrix $A(p)$ is defined in (2.204). The proof is complete. □

2.6 First Order Evolution System of PDE and Cauchy-Kowalevska Theorem

We consider an evolution system of PDE of the following form

$$\begin{cases} \partial_t u_j = & \sum_{i=1}^{n}\sum_{k=1}^{N} a^i_{jk}(t,x,u)\partial_i u_k + b_j(t,x,u),\ j = 1,...,N \\ u(0,x) = & u_0(x) \end{cases} \quad (2.206)$$

where $u = (u_1,...,u_N)$ is an unknown vector function depending on the time variable $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$. The system (2.206) will be called first order evolution system of

PDE and without losing generality we may assume $u_0 = 0$ (see the transformation $\widetilde{u} = u - u_0$) and the coefficients a^k_{ij}, b_j not depend on the variable t(see $u_{N+1} = t$, $\partial_t u_{N+1} = 1$). The notations used in (2.206) are the usual ones $\partial_t u = \frac{\partial u}{\partial t}, \partial_i u = \frac{\partial u}{\partial x_i}$, $i \in \{1, ..., n\}$. Introducing a new variable $z = (x, u) \in B(0, \rho) \subseteq \mathbb{R}^{n+N}$, we rewrite the system (2.206) as follows

$$\begin{cases} \partial_t u_j = & \sum_{i=1}^{n} \sum_{k=1}^{N} a^i_{jk}(z)\partial_i u_k + b_j(z),\ j = 1, ..., N \\ u(0) = & 0 \end{cases} \tag{2.207}$$

Everywhere in this section we assume that the coefficients $a^i_{jk}, b_j \in \mathcal{C}^w(B(0,\rho) \subseteq \mathbb{R}^{n+N})$ are analytic functions, i.e the corresponding Taylor's series at $z = 0$ is convergent in the ball $B(0, \rho) \subseteq \mathbb{R}^{n+N}$ (Brook Taylor 1685-1731)

2.6.1 Cauchy-Kowalevska Theorem

Theorem 2.6.1. *Assume that*

$$a^i_{jk}, b_j \in \mathcal{C}^w(B(o,\rho)) \subseteq \mathbb{R}^{n+N}$$

for any

$$i \in \{1, ..., n\}, k, j \in \{1, ..., n + N\}$$

Then the evolution system (2.207) *has an analytical solution* $u \in \mathcal{C}^w(B(o, a)) \subseteq \mathbb{R}^{n+1}$ *and it is unique with respect to analytic functions.*

Proof. Without restricting generality we assume $n = 1$ and write $a_{jk} = a^1_{jk}$. The main argument of the proof uses the property that the coefficients c^i_{lk} from the associated Taylor's series

$$u_i(t, x) = \sum_{l,k=0}^{\infty} c^i_{lk} t^l x^k = \sum_{l.k=0}^{\infty} \frac{t^l x^k}{l!k!} [\frac{\partial^{k+l}}{\partial t^l \partial x^k} u_i(t, x)]_{t=0,x=0} \tag{2.208}$$

are uniquely determined by the evolution system(2.207) and its analytic coefficients. Then using an upper bound for the coefficients c^i_{lk}, we prove that the series (2.208) is convergent. In this respect we notice that

$$[\frac{\partial^m}{\partial_{x^m}} u_i(t, x)]_{t=0} = 0 \text{ for any } m \geqslant 0 \tag{2.209}$$

(see $u_o(x) = 0$) using (2.207) we find the partial derivatives

$$\frac{\partial u_i}{\partial t}(o,x),\, \frac{\partial^2 u_i}{\partial t \partial x}(o,x),\, \frac{\partial^2 u_i}{\partial t^2}(o,x),\, \frac{\partial^3 u_i}{\partial t^2 \partial x}(o,x),\, \frac{\partial^3 u_i}{\partial x^2}(o,x) \tag{2.210}$$

and so on which are entering in (2.208), for each $i \in \{i, ...N\}$. If

$$a_{jk}(z) = \sum_{|\alpha|=0}^{\infty} g_\alpha^{jk} z^\alpha \,\text{and}\, bj \mid z \mid= \sum_{|\alpha|=0}^{\infty} h_x^j z^\alpha \tag{2.211}$$

for $\mid z \mid \leqslant \rho$, $\alpha = (\alpha_i, ..\alpha_{N+1}), \mid \alpha \mid = \alpha_i + ...\alpha_{N+1}$, and $\alpha_i \geqslant 0, \alpha_i$ integer, the

$$c_{lk}^i = P_{lk}^i[(g_\alpha^{jm})_{\alpha,j,m}, (h_\alpha^j)_{\alpha,j}] \tag{2.212}$$

when P_{lk}^i is a polynomial with positive coefficient($\geqslant 0$). It uses the coefficients e_{lk}^i((2.208)) and noticing that by derivation of a product we get only positive coefficient. Now we are constructing the upper bound coefficient C_{lk}^i for C_{lk}^i given in (2.212) such that

$$\nu_i(t,x) = \sum_{l,k=0}^{\infty} C_{lk}^i t^l x^k,\, i = 1, 2, ..., N \tag{2.213}$$

is a local solution of the Cauchy problem.

$$\partial_t \nu j = \sum_{k=1} N A_{jk}(z) \frac{\partial}{\partial x} \nu(k) + B_j(z),\, \nu_j(0,x) = 0,\, j \in \{1, ..., N\} \tag{2.214}$$

Here A_{jk} and B_j must be determined such that

$$A_{jk}(z) = \sum_{|\alpha|=0}^{\infty} G_\alpha^{jk} z^\alpha,\, B_j(z) = \sum_{|\alpha|=0}^{\infty} H_\alpha^j z^\alpha \tag{2.215}$$

where $\mid g_\alpha^{jk} \mid \leqslant G_\alpha^{jk}$, $\mid h_\alpha^j \mid \leqslant H_\alpha^j$
Using (2.215), we get

$$C_{lk}^i = P_{lk}^i[(G_\alpha^{jm})_{\alpha,j,m}, \mid H_\alpha^j \mid_{\alpha,j}] \geqslant \mid P_{lk}^i[\mid g_\alpha^{jm} \mid_{\alpha,j,m}, (\mid h_\alpha^j)_{\alpha,j}] \mid \geqslant \mid c_{lk}^i \mid \tag{2.216}$$

and the upper bound coefficients $\{C_{lk}^i\}$ are found provided A_{jk} and B_j are determined such that the system(2.214) is satisfied by the analytic solution $v(t,x)$ given in (2.215). In this respect, we define the constants G_α^{jk} and H_α^{jk}andH_α^j using the following estimates

$$M_1 = \max \mid a_{jk}(z) \mid, M = \max \mid b_j(z) \mid j,k = 1, ...N \mid Z \mid \leqslant \rho \tag{2.217}$$

we get

$$\begin{cases} \mid g_\alpha^{jk} \mid \leqslant \frac{M_1}{\rho^{|\alpha|}} \leqslant \frac{M_1}{\rho^{|\alpha|}} \frac{|\alpha|!}{\alpha!} = & G_\alpha^{jk} \\ \mid h_\alpha^j \mid \leqslant \frac{M_2}{\rho^{|\alpha|}} \leqslant \frac{M_2}{\rho^{|\alpha|}} \frac{|\alpha|!}{\alpha!} = & H_\alpha^j \end{cases} \tag{2.218}$$

$$A_{jk}(z) = \sum_{|\alpha|=0}^{\infty} M\frac{|\alpha|!}{\alpha!}(\frac{z}{\rho})^{\alpha} = M\sum_{|\alpha|=0}^{\infty}\frac{1}{\rho^{|\alpha|}}\frac{|\alpha|!}{\alpha!}z^{\alpha}$$
$$= M[1-\frac{z_1+...+z_{N+1}}{\rho}]^{-1}, \text{ for } |z_1|+...+|z_{N+1}|<\rho \quad (2.219)$$

$$B_j(z) = \sum_{|\alpha|=0}^{\infty} H_{\alpha}^{jk} z_{\alpha} = M[1-\frac{z_1+...+z_{N+1}}{\rho}]^{-1}, \text{ for } |z_1|+...+|z_{N+1}|<\rho \quad (2.220)$$

Here we have used

$$\sum_{|\alpha|=k}\frac{k!}{\alpha!}Z^{\alpha} = (\sum_{i=1}^{N+1} z_i)^k,\ \alpha! = (\alpha_1!)...(\alpha_{N+1}!), z^{\alpha} = z_1^{\alpha_1}...Z_{N+1}^{\alpha_{N+1}} \quad (2.221)$$

Notice that the coefficients A_{jk}, B_j do not depend on (j,k) and each component $\nu_i(t,x)$ can be taken equals to $w(t,x)$, $i \in \{1,...,N\}$, where $\{w(t,x)\}$ satisfies the following scalar equation

$$\partial_t w = \frac{M\rho}{\rho - x - Nw}(1+N\partial_x w),\ w(0,x)=0 \text{ for } |x|+N|w|<\rho \quad (2.222)$$

by a direct inspection we notice that for $x\in\mathbb{R}^n$ we need to replace $x\in\mathbb{R}$ in (2.222) by $x \to y = \sum_1^n x_i$ where $x = (x_1,...x_n)$ and $w(t,x) \to w(t,\sum_1^n x_i)$ The solution of (2.222) can be represented by

$$W(t,x) = \frac{1}{2N}[\rho - x - \sqrt{(\rho-x)^2 - 4MN\rho t}],\ if\ |x| < \sqrt{\rho} \quad (2.223)$$

and

$$t < \frac{\rho}{16MN} = T(M,\rho)$$

In the case $x\in\mathbb{R}^n$ we use $y = (\sum_1^n x_i)$ instead of $x\in\mathbb{R}$ and $w(t,y)$ satisfy both the equation (2.222) and the representation formula(2.223). □

2.6.2 1st Order Evolution System of Hyperbolic & Elliptic Equations

In the following we shall rewrite some hyperbolic and elliptic equation appearing in Mathematical Physics as an evolution system of first order equation. It suggest that assuming only analytic cofficients we may and do solve these equations applying the Cauchy-Kowalevska theorem (C-K)
(E_1). **Klein-Gordon equations(D.B Klein 1894-1977, W.Gordon 1893-1939)**
They are describing the evolution of a wave function y associated with a **vanishinq**

"spin" particle and in differential form is expressed using standard notations, as follows

$$\begin{cases} \partial_t^2 y = \Delta y - my + f(y, \partial_t y, \partial_x y),\ m > 0, t \geqslant 0, y \in \mathbb{R} \\ y(0,x) = y_0(x) = \partial_t y(0,x) = y_1(x),\ x \in \mathbb{R}^n \end{cases} \tag{2.224}$$

where $\partial_t y = \frac{\partial y}{\partial t}, \partial_x y = (\partial_1 y, ..., \partial_n y),\ \partial_i y = \frac{\partial y}{\partial x_i}$ and
the laplacian $\Delta y = \sum_1^n \partial_i^2 y$. Denote

$$u = (\partial_x y, \partial_t y, y) \in \mathbb{R}^{n+2},\ u^0(x) = (\partial_x y_0(x), y_1(x), y_0(x))$$

and $F(u) = f(y, \partial_t y, \partial_x y)$. Then (2.224) is written as an evolution system

$$\begin{pmatrix} \partial_t u_j = & \partial_j u_{n+1},\ j \in \{1, ..., n\} \\ \partial_t u_{n+1} & \sum_{i=1}^n \partial_i u_i + F(u) \\ \partial_t u_{n+2} = & u_{n+1} \\ u(0,x) = & u^0(x) \end{pmatrix} \tag{2.225}$$

(E_2). **Maxwell equations(J.C.Maxwell 1831-1879)**
Denote the electric field $E(t,x) \in \mathbb{R}^3, x \in \mathbb{R}^3, t \in \mathbb{R}$, the magnetic field $H(t,x) \in \mathbb{R}^3, x \in \mathbb{R}^3, t \in \mathbb{R}$ and the Maxwell equations are the following

$$\begin{pmatrix} \frac{\partial \epsilon(E)}{\partial E} \partial_t E - \Delta \times H = & 0, E(0,x) = E^0(xs) \\ \frac{\partial \mu(H)}{\partial H} \partial_t H + \Delta \times E = & 0, H(0,x) = H^0(x) \end{pmatrix} \tag{2.226}$$

where $D = \epsilon(E)$ and $B = \mu(H)$ are some nonlinear mappings satisfying

$$\epsilon(E), \mu(H) : \mathbb{R}^3 \to \mathbb{R} \tag{2.227}$$

are analytic functions with uniformly positive definite matrices $\frac{\partial \epsilon(E)}{\partial E}, \frac{\partial \mu(H)}{\partial H}$ for E, H in bounded sets. In addition, we assume

$$\begin{aligned} \epsilon(E) &= \epsilon_0 E + O(\mid E \mid^3),\ \mu(H) \\ &= \mu_0 H + O(\mid H \mid^3)\, with\, \epsilon_0 > 0, \mu_0 > 0 \end{aligned} \tag{2.228}$$

$$\epsilon(E) = \epsilon_0 E + O(\mid E \mid^3),\ \mu(H) = \mu_0 H + O(\mid H \mid^3) \text{ with } \epsilon_0 > 0, \mu_0 > 0 \tag{2.229}$$

$$\text{div} D(t,x) = 0, \text{div} B(t,x) = 0, D(0,x) = D^0(x), B(0,x) = B^0(x) \tag{2.230}$$

where $D(t,x) = \epsilon(E(t,x))$ and $B(t,x) = \mu(H(t,x))$. The vectorial products $\Delta \times H$ and $\Delta \times E$ are given formally by the corresponding determinant

$$\Delta \times E = det \begin{pmatrix} i & j & k \\ \partial_1 & \partial_2 & \partial_3 \\ H_1 & H_2 & H_3 \end{pmatrix} = i(\partial_2 H_3 - \partial_3 H_2) + j(\partial_3 H_1 - \partial_1 H_3) + k(\partial_1 H_2 - \partial_2 H_1) \tag{2.231}$$

$$\Delta \times E = det \begin{pmatrix} i & j & k \\ \partial_1 & \partial_2 & \partial_3 \\ E_1 & E_2 & E_3 \end{pmatrix} = i(\partial_2 E_3 - \partial_3 E_2) + j(\partial_3 E_1 - \partial_1 E_3) + k(\partial_1 E_2 - \partial_2 E_1) \tag{2.232}$$

where $(i, j, k) \in \mathbb{R}^3$ is the canonical basis. Denote $u = (E, H) \in \mathbb{R}^6$ and the system (2.226) for which the assumptions (2.227), (2.229) and (2.230) are valid will be written as an evolution system

$$\partial_t u = \sum_{j=1}^{3} [A^0(u)]^{-1} A^j \partial_j u,\ u(0, x) = u^0(x) = (E^0(x), H^0(x)) \tag{2.233}$$

where (6×6) matrices A^i are given by

$$A^0(u) = \begin{pmatrix} \frac{\partial \epsilon(E)}{\partial E} & O_3 \\ O_3 & \frac{\partial \mu(H)}{\partial H} \end{pmatrix},\ A^1 = \begin{pmatrix} & & & 0 & 0 & 0 \\ & O_3 & & 0 & 0 & -1 \\ & & & 0 & 1 & 0 \\ 0 & 0 & 0 & & & \\ 0 & 0 & 1 & & O_3 & \\ 0 & -1 & 0 & & & \end{pmatrix}$$

$$A^2 = \begin{pmatrix} & & & 0 & 0 & 1 \\ & O_3 & & 0 & 0 & -0 \\ & & & -1 & 0 & 0 \\ 0 & 0 & -1 & & & \\ 0 & 0 & 1 & & O_3 & \\ 1 & 0 & 0 & & & \end{pmatrix},\ A^3 = \begin{pmatrix} & & & 0 & -1 & 0 \\ & O_3 & & 1 & 0 & 0 \\ & & & 0 & 0 & 0 \\ 0 & 1 & 0 & & & \\ -1 & 0 & 1 & & O_3 & \\ 0 & 0 & 0 & & & \end{pmatrix}$$

where O_3 is (3×3) zero matrix, $A^0(u)$ is strictly positive definite matrix and each A^i is a symmetric matrix. Recalling that Maxwell equation means the evolution system with analytic coefficients (2.233) and, in addition the constraints (2.230), we notice that (2.230) are satisfied, provide

$$\text{div} D^0(x) = 0 \text{ and } \text{div} B^0(x) = 0 \tag{2.234}$$

In this respect, we rewrite the original system (2.226) as

$$\partial_t D \Delta \times H = 0,\ \partial_t B + \Delta \times E0,\ D(0, x) = D^0(x),\ B(0, x) = B^0(x) \tag{2.235}$$

and by a direct computation we get

$$\begin{cases} \partial_t[\text{div} D(t, x)] = \text{div}[\partial_t D](t, x) = \text{div}(\Delta \times H)(t, x) = 0 \\ \partial_t[\text{div} B(t, x)] = div[\partial_t B](t, x) = \text{div}(\Delta \times E)(t, x) \end{cases} \tag{2.236}$$

which show that (2.234) implies the constraints (2.230)

2.7 (E_3) Plate Equations

(they cannot be solved by (C-K) theorem). They are described by the following equation

$$\partial_t^2 y + \Delta^2 y = f(\partial_t y, \partial_x^2 y) + \sum_{i=1}^{n} b_i(\partial_t y, \partial_x^2 y)\partial_i(\partial_t y),\ x \in \mathbb{R}^n \tag{2.237}$$

where f and b_i are analytic functions, and $y(t,x) \in \mathbb{R}$ satisfies Cauchy conditions.

$$y(0,x) = y_0(x),\ \partial_t y(0,x) = y_1(x) \tag{2.238}$$

Here "Δ"is the standard laplacian operator, "∂_x"is the gradient acting as linear mappings for which "Δ^2"and "∂_x^2"have the usual meaning. Denote

$$u = (\partial_x^2 y, \partial_x(\partial_t y),\ \partial_t y) = (u_1, \ldots, u_{n^2}, u_{n^2+1}, \ldots, u_{n^2+n}, u_{n^2+n+1})$$

$$= (y_{11}, y_{12}, \ldots, y_{n1}, \ldots, y_{1n}, y_{2n}, \ldots, y_{nn}, u_{n^2+1}, \ldots, u_{n^2+n}, u_{n^2+n+1}) \in R^{n^2+n+1}$$

as the unknown vector function, where $(n \times n)$ matrix $\partial_x^2 y$ is denoted as a rowvector $(y_{11}, \ldots, y_{nn}) \in R^{n^2}$. By a direct inspection we see that

$$\begin{cases} \partial_t y_{ij} = \partial_i \partial_j {u_n}^2 + n + 1 = \partial_i u_{n^2+j},\ i,j \in \{1, \ldots, n\} \\ \partial_t u_{n^2+n+1} = -\sum\limits_{i=1}^{n} \triangle(y_{ii}) + F(u) \\ \partial_t \partial_t u_{n^2+j} = \partial_j(\partial_t u_{n^2+n+1}) = -\sum\limits_{i=1}^{n} \partial_j \triangle(y_{ii}) + \sum\limits_{k=1}^{n^2+n+1} \frac{\partial}{\partial u_k} F(u) \partial_j u_k \\ u(0,x) = (\partial_x^2 y_0(x), \partial_x,\ y_1(x), y_1(x)) = u^0(x) \end{cases} \tag{2.239}$$

where $F(u) = f(u_{n^2+n+1}, \partial_x^2 y) + \sum\limits_{i=1}^{n} b_i(u_{n^2+n+1}, \partial_x^2 y) u_{n^2+i}$. As far as the system (2.239) contains laplacian "Δ"in the right hand side it cannot be asimilated with a first order evolution system. Actually, it is well known that a parabolic equation has no rewriting as a first order evolution system and show that the simplest plate equations.

$$\partial_t^2 y + \Delta^2 y = 0\ \ y(0,x) = y_0(x),\ \partial_t y(0, x = y_1(x)) \tag{2.240}$$

are equivalent with a parabolic type equation (Schrödinger)

$$\begin{cases} \partial_t w = i\Delta w (or\ - i\partial_t w(t,x) = \Delta_x w(t,x)) \\ w(o,x) = y_1(x) + i\Delta y_0(x) = w^0(x) \end{cases} \tag{2.241}$$

Remark 2.7.1. *Let $\{y(t,x)\}$ be the analytic solution of the plate equation* (4)*. Define the analytic function $w(t,x) = \partial_t y(t,x) + i\Delta_x y(t,x) \in \mathbb{C}$. Then $\{w(t,x) : (t,x) \in D \subseteq \mathbb{R}^{n+1}\}$ satisfies Schrödinger (S) equation* (2.241) *and initial condition $w(o,x) = y_1(x) + i\Delta y_0(x) = w^0(x)$. Conversely, if $w(t,x) = y(t,x) + iz(t,x)$ satisfies* (2.241)

then $Rew(t,x)\,4 = y(t,x)$ *and* $Imw(t,x) = z(t,x)$ *are real analytic solution for the homogeneous equation of the plate*(2.240) *with Cauchy conditions* $y(0,x) = y_0(x)$ *and* $z(0,x) = z_0(x)$ *correspondingly. Moreover, the solution of the Schrödinger equation* (2.241) *agrees with the following representation* $w(t,x) = u(it,x)$ *where* $u(\tau,x) = (4\pi\tau)^{\frac{n}{2}} \int_{\mathbb{R}^n} [exp(-)]u_0(y)dy$ *and we get* $-i\partial_t w(t,x) = \partial_\tau u(it,x) = \Delta_x u(it,x) = \Delta_x w(t,x)$ *(S-equation).*

2.7.1 Exercises

(1) Using the algorithm for Klein-Gordon equation, write the evolution system corresponding to the following system of hperbolic equations

$$\partial_t^2 y_i = \sum_{m,j,k=1}^{n} C_{imjk}(\partial_x y)\partial_m \partial_k y_j,\ \ i = 1,...,n\,, y = (y_1,...,y_n)$$

$$y_i(0,x) = y_i^0(x)\,,\ \partial_t y_i(0,x) = y_i^1(x)\,,\ i = 1,...,n\,, x \in \mathbb{R}^n$$

(it appears in elasticity field)
Hint. $\partial_x y = (\partial_x y_1, ..., \partial_x y_n)\,,\ \partial_x y_i = (\partial_1 y_i, ..., \partial_n y_i)\,, j \in \{1,...,n\}$
Denote $u_{ik} y = \partial_k y_i(t,x)\,,\ k \in \{1,...,n\}\,,\ i \in \{1,...,n\}$

$$u_{n^2+i} = \partial_t y_i\,,\ i \in \{1,...,n\}$$

$$u_{n^2+n+i} = y_i\,,\ i \in \{1,...,n\}$$

we get

$$\begin{cases} \partial_t\, u_{ik} = \partial_k \partial_t\, y_i = \partial_{x_k}\, u_{n^2+i}\ \ i \in \{1,...,n\} \\ \partial_t\, u_{n^2+i} = \partial_t^2 = \sum\limits_{m,j,k=1}^{n} C_{imjk}(u_{11},...,u_{nn})\partial_{x_m}(u_{kk})\,,\ i \in \{1,...,n\} \\ \partial_t u_{n^2+n+i} = \partial_t\, y_i = u_{n^2+i}\,,\ i \in \{1,...,n\} \end{cases}$$

(2) Using (C-K) theorem solve the following elliptic equation

$$\partial_x^2 + \partial_y^2 = x^2 + y^2\,, u(0,y) = \partial_x\, u(0,y) = 0$$

Hint. Denote $t = x, \partial_y u = u_1,\ \partial_x u = u_2\,, u = u_3$ and we get

$$\begin{cases} \partial_t u_1 = \partial_y(\partial_t u) = \partial_y u_2 \\ u_1(0,y) = 0 \end{cases} , \begin{cases} \partial_t u_2 = \partial_x^2 u = -\partial_y u_1 + t^2 + y^2 \\ u_2(0,y) = 0 \end{cases} , \begin{cases} \partial_t u_3 = u_2 \\ u_3(0,y) = 0 \end{cases}$$

We are looking for

$$u_3(t,y) = u_3(0,y) + \frac{t}{1!}\partial_t u_3(0,y) + \frac{t^2}{2!}\partial_t^2 u_3(0,y) + ...$$

Now compute

$$u_3(0,y)=0\,,\partial_t u_3|_{t=0}=u_2|_{t=0}=0\,,\ \partial_t^2 u_3|_{t=0}=\partial_t u_2|_{t=0}=y^2\,,\ \partial_t^k u_3|_{t=0}$$

for $k\geqslant 3$. We obtain $u(x,y)=u_3(x,y)=\frac{1}{2}x^2y^2$
(3) by the same method solve

$$\partial_x^2+\partial_y^2=y^2\,,u(0,y)=\partial_x\,u(0,y)=0$$

and get

$$u(x,y)=\frac{x^2y^2}{2}-\frac{x^4}{12}$$

2.7.2 The Abstract Cauchy-Kawalewska Theorem

(L.Nirenberg, J.Diff.Geometry 6(1972)pp 561-576)
For $0<s<1$, let X_s be the space of vectorial functions $\nu(x)\in\mathbb{C}^n$ which are holomorphic and bounded on $D_s=\prod_{j=1}^{\alpha}\{\mid x_j\mid< sR\}\subseteq\mathbb{C}^d$. Denote $\parallel\nu\parallel_s=\sup_{D_s}\mid\nu(x)\mid$. Then X_s is a Banach space and the natural inspection $X_s\subseteq X_{s'}(s'\leqslant s)$ has the norm $\parallel i\parallel\leqslant 1$. By the standard estimates of the derivatives (see Cauchy formula), we get $\parallel\partial_j\nu\parallel_{s'}\leqslant R^{-1}\parallel\nu\parallel_s (s-s')$ for any $0<s'<s<1$ and for a linear equation the following holds true

Theorem 2.7.2. *Let $A(t):X_s:\to X_{s'}$ be linear and continuous of $\mid t\mid<\eta$ for any $0<s'<1$ fulfilling*

$$(i_1)\parallel A(t)v\parallel_{s'}\leqslant C\parallel v\parallel_{s-s'},\ \forall\, 0<s'<s<1$$

Let $f(t)$ be a continuous mapping of $\mid t\mid<\eta$ with values in X_s, $0<s<1$ fulfilling

$$(i_2)\parallel f(t)\parallel_s\leqslant\frac{K}{a(1-s)},\ for\, 0<s<1$$

where $0<a<\frac{1}{8K}$ is fixed and $K>0$ is given constant. Then there exists a unique function $u(t)$ which is continuously differentiable of $\mid t\mid<a(1-s)$ with values in X_s for each $0<s<1$, fulfilling

$$(C_1)\frac{du}{dt}(t)=\ A(t)u(t),u(0)=0$$

$(C_2)\parallel u(t)\parallel_s\leqslant 2K(\frac{a(1-s)}{|t|}-1)^{-1}$ *for any* $\mid t\mid<a(1-s)$.

Theorem 2.7.3. *The nonlinear case is refering to the following Cauchy problem*

$$(\alpha)\frac{du}{dt}=F(u(t),t)\ \mid t\mid<\eta,\ u(0)=0$$

Let the condition (α_1) *and*(α_5) *be fulfilled. Then there exists* $a > 0$ *and a unique function* $u(t)$ *which is continuously differentiable of* $\mid t \mid< a(1-s)$ *with values in* X_s *fulfilling the equation* (α) *and* $\parallel u(t) \parallel_s < R\ \forall\ \mid t \mid< a(1-s)$*, for each* $0 < s < 1$.

2.8 Appendix Infinitesimal Invariance

J.R.Olver (Application 0f Lie algebra to Diff.Eq, Springer, 1986, Graduate texts in mathematics; 107)

1. One can replace the complicated, nonlinear conditions for the invariance of a subset or function under a group of transformations by an equivalent linear condition of infinitesimal invariance under the corresponding infinitesimal generators of the group action. It will provide the key to the explicit determination of the symmetry groups of systems of differential equations. We begin with the simpler case of an invariant function under the flow generated by a vector field which can be expressed as follows

$$f(G(t;x)) = f(x),\, t \in (-a,a)\, x \in D \subseteq \mathbb{R}^n \Longleftrightarrow \tag{2.242}$$

$$g(f)(x) =< \partial_x f(x), g(x) >= 0,\, \forall\, x \in D. \tag{2.243}$$

where $G(0;x) = x$, $\frac{dG(t;x)}{dt} = g(G(t;x))$, $t \in (-a,a)$, $x \in D$

Theorem 2.8.1. *Let* G *be a group of transformation acting on a domain* $D \subseteq \mathbb{R}^n$*. Let* $F : D \to \mathbb{R}^m$*,* $m \leqslant n$*, define a system of algebraic equations of maximal rank*

$$(F_j(x) = 0,\, j = 1,...,m)\,; rank \frac{\partial F}{\partial x)} x) = m,\, \forall\, x \in D, F(x) = 0 \tag{2.244}$$

Then G *is a symmetry group of system* iff

$$g(F_j)(x) = 0,\, \forall\, x \in \{y \in D : F(y) = 0\},\, j = 1,2,...,m \tag{2.245}$$

where g *is the infinitesimal generators of* G.

Proof. The necessity of (2.245) follows by differentiating the identity $F(G(t;x)) = 0$, $t \in (-a,a)$ in which x is solution of(2.244) and $G(t;x)\,, t \in (-a,a)$, is the flow generated by the vector field g. To prove the sufficiency, let x_0 be a solution of the system using the maximal rank condition we can choose a coordinate transformation $y = (y^1,...,y^n)$ such that $x_0 = 0$ and F has the simple form $F(y) = (y^1,...,y^m)$. Let $g(y) = (g^1(y),...,g^n(y)) \in \mathbb{R}^n$ be any infinitesimal generators of G expressed in the new coordinates and rewrite (2.245) as follows

$$g^i(y) = 0\ , i = 1,2,...,m,\, \forall\, y \in \mathbb{R}^n,\, y^1 = ... = y^m = 0 \tag{2.246}$$

Now the flow $G(t)(x_0)$, $t \in (-a,a)$ generated by the vector field g and passing through $x_0 = 0$ satisfies the system of ordinary differential equations $\frac{dG^i}{dt}(t)(x_0) =$

$g^i(G(t)(x_0))$, $G(0)(x_0) = 0$, $G^i(t)(x_0) = 0$, $t \in (-a, a)$, $i = 1, ..., m$ which means $F(G(t)(x_0)) = 0$, $t \in (-a, a)$. The proof is complete. $\square$

Example 2.8.1. *Let $G = SO(2)$ be the rotation group in the plane, with infinitesimal generator ($g = -y\partial_x + x\partial_y$)*
$g(x, y) = col(-y, x)$. The unit circle $S_1 = \{x^2 + y^2 = 1\}$ is an invariant subset of $SO(2)$ as it is the solution set of the invariant function $f(x, y) = x^2 + y^2 - 1$. Indeed, $g(f)(x, y) = -2xy+2xy = 0$, $\forall(x, y) \in \mathbb{R}^2$ so the equations (4) are satisfied on the unit circle itself. The maximal rank condition does hold for f since its gradient $\partial f(x, y) = col(2x, 2y)$ does not vanish on S^1. As a less trivial example,consider the function $f(x, y) = (x^2 + 1)(x^2 + y^2 - 1)$ and notice that $g(f)(x, y) = -2xy(x^2 + 1)^{-1}f(x, y)$ which shows that $g(f)(x, y) = 0$ whenever $f(x, y) = 0$. In addition $\partial f(x, y) = (4x_3 + 2xy^2, 2x^2y + 2y)$ vanishes only when $x = y = 0$ which is not a solution to $f(x, y) = 0$. We conclude that the solution set $\{(x, y) : (x^2+1)(x^2+y^2-1) = 0\}$ is a rotationally-invariant subset of $\mathbb{R}^2$.

Remark 2.8.2. *For a given one-parameter group of transformations $y = G(t)(x)$, $x \in D \subseteq \mathbb{R}^n$, $t \in (-a, a)$ generated by the infinitesimal generator*

$$\overrightarrow{g} = g^1(x)\partial_{x^1} + ... + g^n(x)\partial_{x^n} (see\ \frac{dG(t)(x)}{dt} = g(G(t)(x)),\ t \in (-a, a), G(0)(x) = x$$

$g(x) = col(g^1(x), ..., g^n(x))$. We notice that the corresponding invariant functions are determined by the standard first integrals associated with the ODE $\frac{dx}{dt} = g(x)$.

2.8.1 Groups and Differential Equations

Suppose we are considering a system S of differential equation involving p independent variables $x = (x^1, ..., x^p)$ and q dependent variables$u = (u^1, ..., u^q)$. The solution of the system will be of the form $u^\alpha = f^\alpha(x)$, $\alpha = 1, ..., q$. Denote $X = \mathbb{R}^p, U = \mathbb{R}^q$ and a symmetry group of the system S will be a local group of transformations,G, acting on some open subset $M \subseteq X \times U$ in such a way that" G transforms solutions of S to other solutions of S". To proceed rigorously, define the graph of $u = f(x)$,

$$\Gamma_f = \{(x, f(x)) : x \in \Omega\} \subseteq X \times U$$

where $\Omega \subseteq X$ is the domain of definition of f. Note that Γ_f is a certain $p-$dimentional submanifold of $X \times U$. If $\Gamma_f \subseteq M_G$(domain of definition of the group transformations G) then the transform of Γ_f by G is just

$$G.\Gamma_f = \{(\widehat{x}, \widehat{u}) = G(x, u) : (x, u) \in \Gamma_f\}$$

The set $G\Gamma_f$ is not necessarily the graph of another single valued function $\widehat{u} = \widehat{f}(\widehat{x})$ but if it is the case then we write $\widehat{f} = G.f$ and $\widehat{f}$ the transform of f by G.

2.8.2 Prolongation

The infinitesimal methods for algebraic equations can be extended for "systems of differential equations". To do this we need to prolong the basic space $X \times U$ to a space which also represents the various partial derivatives occurring in the system.If $f : X \to U$ is a smooth function,$u = f(x) = (f^1(x), ..., f^q(x))$ there are $q.p_k$ numbers $u_j^\alpha = \partial_j f^\alpha(x)$ needed to represent all different $k-th$ order derivatives of the components of f at point x. We let $U_k = \mathbb{R}^q.p_k$ be the Euclidean space of this dimension,endowed with coordinates u_j^α corresponding to $\alpha = 1, ..., q$, and all multi-indices $J = (j_1, ..., j_k)$ of order k, designed so as to represent $\partial_J f^\alpha(x) = \frac{\partial^k f^\alpha(x)}{\partial_x^{j_1} \partial_{x^{j_k}}}$.
Set $U^n = U \times U_1 \times ... \times U_n$ to be the product space.whose coordinates represents all the derivatives of functions $u = f(x)$ of all orders from 0 to n. Note that U^n is Euclidean space of dimension

$$q + qp_1 + ... + qp_n = qp^n$$

Atypical point in U^n will be denoted by u^n so u^n has $q.p^n$ different components u_j^α where $\alpha = 1, ..., q$ has J sums over all unordered multi-indices $J = (j_1, ..., j_k)$ with $1 \leqslant j_k \leqslant p, j_1 + ... + j_k = k$ and $0 \leqslant k \leqslant n$. (By convention for $k = 0$ there is just one such multi-index,denoted by O, and u_0^α just replace to the component u^α of u itself)

Example 2.8.2. *$p = 2, q = 1$.Then $X = \mathbb{R}^2$, $(x^1, x^2) = (x, y)$ and $U = \mathbb{R}$ has the single coordinate u. The space U_1 isomorphic to $\mathbb{R}^2$ with coordinates (u_x, u_y) since these represents all the first order partial derivatives of u with respect to x and y. Similarly $U_2 = \mathbb{R}^3$ has coordinates (u_{xx}, u_{xy}, u_{yy}) representing the second order partial derivatives of u, namely $\frac{\partial^2 u}{\partial x^i \partial y_{2-i}}$, $i = 0, 1, 2$. In general, $U_k = \mathbb{R}^{k+1}$, since there are (k+1) k-th order partial derivatives of u, namely $\frac{\partial^k u}{\partial x^i \partial y_{k-i}}$, $i = 0, 1, .., k$. Finally, the space $U^2 = U \times U_1 \times U_2 = \mathbb{R}^6$, with coordinates $u^2 = (u; u_x, u_y, u_{xx}, u_{xy}, u_{yy})$ represents all derivatives of u with respect tox and y of order at most 2. Given a smooth function $u = f(x)\, f : X \to U$ there is an induced function $u^n = pr^{(n)} f(x)$ called the $n-th$ prolongation of f which is defined by the equations*

$$u_j^\alpha = \partial_J f^\alpha(x),\ J = (j_1, ..., j_k),\ 1 \leqslant j_k \leqslant p,\ j_1 + ... + j_k = k,\ 0 \leqslant k \leqslant n$$

*for each $\alpha = 1, \ldots, q$ (see $X = \mathbb{R}^p$, $U = \mathbb{R}^q$).
The total space $X \times U^n$ whose coordinates represents the independent variables,the dependent variables and the derivatives of the dependent variables up to order n is called the $n-th$ order jet space of the underlying space $X \times U$ (it comes from viewing $pr^{(n)} f$ as a corresponding polynomial degree n associated with its Taylor series ar the point x)If the differential equations are defined in some open subset $M \subset X \times U$ then we define the n-jet space $M^{(n)} = M \times U_1 \times, \dot{,} \times U_n$ of M.*

2.8.3 Systems of Differential Equations

A system S of $n-th$ order differential equation in p independent and q dependent variables is given as a system of equations

$$\Delta_\nu(x, u^{(n)}) = 0, nu = 1, \ldots, b \tag{2.247}$$

involving $x = (x^1, \ldots, x^p), u = (u^1, \ldots, u^q)$ and the derivatives of u with respect to x up to order n. The function $\Delta(x, u^n) = (\Delta_1(x, u^n)), \ldots, \Delta_l(x, u^n)$, will be assumed to be smooth in their arguments so Δ can be viewed as a smooth map from the jet space $X \times U^n$ to some $l-$ dimensional Euclidean space $\Delta : X \times U^n \to \mathbb{R}^l$.
The differential equations themselves tell where the given map Δ variables on $X \times U^n$ and thus determine a subvariety

$$S_\Delta = \{(x, u^n) : \Delta(x, u^n) = 0\} \subseteq X \times U^n \tag{2.248}$$

0n the total jet space.
From this point of view, a smooth solution of the given system of differential equations is a smooth function $u = f(x)$ such that

$$\Delta_\nu(x, pr^{(n)} f(x)) = 0, \nu = 1, ..., l$$

whenever x lies in the domain of f. This condition is equivalent to the statement that the graph of the prolongation $pr^{(n)} f(x)$ must lies entirely within the subvariety S_Δ determined by the system

$$\Gamma_f^{(n)} = \{(x, pr^{(n)} f(x))\} \subseteq S_\Delta = \{\Delta(x, u^{(n)}) = 0\} \tag{2.249}$$

We can thus take an $n-th$ order system of differential equations to be a subvariety S_Δ in the $n = jet$ space $X \times U^n$ and a solution to be a function $u = f(x)$ such that the graph of the $n-th$ prolongation $pr^{(n)} f$ is contained in the subvariety S_Δ.

Example 2.8.3. *Consider Laplace equation in the plane*

$$u_{xx} + u_{yy} = 0 \tag{2.250}$$

Here $p = 2, q = 1, n = 2$ *coordinates* $(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy})$ *of* $X \times U^n$ *(a hyperplane)there, and this is the set* $_\Delta$ *for Laplac's equation.A solution must satisfy*

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = o \ \ \forall (x, y)$$

This is clearly the same as requiring that the graph of the second prolongation $pr^2 f$ *lie in* S_Δ*. For example, if*

$$f(x, y) = x^3 - 3xy^2$$

then

$$pr^2 f(x, y) = (x^3 - 3xy^2; 3x^2 - 3y^2, -6xy; 6x, -6y, -6x)$$

which lies in

$$S_{\Delta}(\text{ see } 6x + (-6x) = 0)$$

2.8.4 Prolongation of Group Action and Vector Fields

We begin by considering a simple first order scalar differential equation

$$\frac{du}{dx} = F(x, u), \quad (x, u) \in D \subseteq \mathbb{R}^2 \tag{2.251}$$

This condition is invariant under a one-parameter group of transformations $G(\epsilon) = \exp \epsilon g$ defined on an open subset $D \subseteq \times U = \mathbb{R}^2$ where

$$g = \xi(x, u)\partial_x + \phi(x, u)\partial_u \tag{2.252}$$

is the infinitesimal generator.
Let $u = u(x; x_0, u_0)$ be s solution of (2.251) satisfying $u(x; x_0, u_0) = u_0$ and $(x_0, u_0) \in D$ arbitrarily fixed. Then $u(\widehat{x_0}(\epsilon); x_0, u_0) = \widehat{u_0}(\epsilon), \epsilon \in (-a, a)$
where

$$(\widehat{x_0}(\epsilon), \widehat{u_0}(\epsilon)) = G(\epsilon)(x_0, u_0)$$

It leads us to the following equations

$$\frac{du}{dx}(\widehat{x_0}(\epsilon)).\frac{d\widehat{x_0}}{d\epsilon}(\epsilon) = \frac{d\widehat{u_0}}{d\epsilon}(\epsilon) \Longleftrightarrow F(x_0, u_0).\xi(x_0, u_0) = \phi(x_0, u_0) \tag{2.253}$$

for any $(x_0, u_0) \in D \subseteq \mathbb{R}^2$.
On the other hand, the equation (2.251) can be viewed as an algebraic constraint

$$F(x, u) - u_x = 0 \tag{2.254}$$

on the variables $(x, u, u_x) \in \mathbb{R}^3$ and we may do ask to find a prolonged and parameter group of transformations $G(1)(\epsilon) + \exp \epsilon g^{(1)}$ acting on a prolonger subvariety $M^{(1)} = X \times U \times U^{(1)} = \mathbb{R}^3$ such that the set solution (2.254)is invarient. In this respect, notice that the new vector field $g^{(1)}$ is a prolongation of the vector field g given in (2.252), $pr^{(1)}g = g^{(1)}$

$$g^{(q)}(x, u, u_x) = \xi(x, u)\partial_x + \phi(x, u)\partial_u + \eta(x, u)\partial_{u_x} \text{where}(\xi(x, u), \phi(x, u),) \tag{2.255}$$

satisfies (2.253) $\forall (x, u) \in D \subseteq R^2$. The set solution(2.254) is invariant under the infinitesimal generator (2.255) $iff(f(x, u, u_x) = F(x, u) - u_x)$

$$(\partial_x f)\xi + (\partial_u f)\phi + (\partial_{u_x} f)\eta = 0 \tag{2.256}$$

for any (x, u, u_x) verifying (2.254). An implicit computation of (2.256) show us that $(\eta(x, u, u_x), \phi(x, u), \xi(x, u))$ must satisfy the following first order partial differential

equation

$$\eta(x,u,u_x) = \partial_x\phi(x,u) + [\partial_u\phi(x,u) - \partial_x\xi(x,u)]u_x - \partial_u\xi(x,u).u_x^2 \tag{2.257}$$

Once we found a symmetry group G, the integration of the equation (2.251) may become a simplex one using elementary operations like integration of a scalar function.

2.8.5 Higher Order Equation

Consider a single $n-th$ order differential equation involving a single dependent variable u

$$\Delta(x,u^{(n)}) = \Delta(x,u,\frac{du}{dx},\ldots,\frac{d^nu}{dx^n} = 0\ x \in \mathbb{R} \tag{2.258}$$

If we assume that(2.258) does not depend either of u or x then the order of the equation can be reduced by one. In this respect consider that Δ in (2.258) satisfied $\partial_u\Delta = 0 i.e$ the vector field $g = \partial_u$ has a trivial prolongation $pr^n g = g = \partial_u$ generating a corresponding prolonged group of symmetry $G^{(n)}$ for the equation

$$\widetilde{\Delta}(x,u_1,...u_n = 0)\,,\text{where}\, u_i = \frac{d^iu}{dx^i},\ i \in \{1,\ldots,n\}1,...n \tag{2.259}$$

Denote $z = \frac{du}{dx}$ and rewrite (2.259) as

$$\widehat{\Delta}(x,z,\frac{dz}{dx},\ldots,\frac{d^{n-1}z}{dx}) = \widetilde{\Delta}(x,z^{(n-1)}) = 0 \tag{2.260}$$

Whose solutions provide the general solution for (2.259) and $u(x) = \int_0^x h(y)dy + c$ is a solution for (2.259) provided $z = h(x)$ is a solution for (2.260). The second elementary group of symmetry for (2.258) is obtained assuming that Δ does not depend on x and write(2.258) as

$$\widetilde{\Delta}(u^{(n)}) = \widetilde{\Delta}(u,\frac{du}{dx},\ldots,\frac{d^nu}{dx^n}) = 0 \tag{2.261}$$

This equation is clearly invariant under the group of transformations in the $x-$direction, with infinitesimal generator $g = \partial_x$. In order to change this into the vector field $g = \partial_\nu$, corresponding to translations of the dependent variable,it suffices to reverse the rules of dependent and independent variable; we set $y = u$, $\nu = x$. Then compute $\frac{du}{dy} = \frac{1}{\frac{d\nu}{dy}}$, using $u(\nu(y)) = y$. Similarly, we get

$$\frac{d^2u}{dx^2} = -\frac{\nu'' y}{(\nu' y)^3},\ldots,\frac{d^nu}{dx^n} = \delta_n(\nu_y^{(1)},\ldots,\nu_y^{(n)})$$

and rewrite (2.261) as follows

$$\widetilde{\Delta}(u^{(n)}) = \widetilde{\Delta}(u,\frac{du}{dx},\ldots,\frac{d^nu}{dx^n}) = \widehat{\Delta}(y,\nu_y^{(1)},\ldots,\nu_y^{(n)}) = \Delta \tag{2.262}$$

where $\nu_y^{(k)} = \frac{d^k \nu}{dy^k}$, $k = 1, \ldots, n$. The equation (2.262) is transformed into a $(n-1)-th$ order differential equation as above denoting $\nu_y^{(1)} = z$ as the new unknown function.

Bibliographical Comments

The first part till to Section 2.3 is written following the references [8],[12] and [13]. Section 2.3 contained in a ASSMS preprint (2010). Sections 2.4 and 2.5 are written following the book in the reference [11]. Sections 2.6 and 2.7 are using more or less the same presentation as in the reference [12].

Chapter 3

Second Order Partial Differential Equations

3.1 Introduction

PDE of second order are written using symbols $\partial_t, \partial_x, \partial_y, \partial_z, \partial_t^2, \partial_x^2, \partial_y^2, \partial_z^2$ where $t \in \mathbb{R}$ is standing for the time variable, $(x, y, z) \in R^3$ are space coordinates, and a symbol $\partial_s(\partial_s^2)$ represent first partial derivative with respect to $s \in \{t, x, y, z\}$ (second partial derivative). There are three types of second order PDE we are going to analyze here and they are illustrated by the following examples

$$\begin{cases} \partial_x^2 u - \partial_y^2 u = & 0 & \text{(hyperbolic)} & (u(x,y) \in \mathbb{R}, (x,y) \in \mathbb{R}^2) \\ \triangle u = \partial_x^2 u + \partial_y^2 u = & 0 & \text{(elliptic)} & (u(x,y) \in \mathbb{R}, (x,y) \in \mathbb{R}^2) \\ \partial_t u = \partial_x^2 u + \partial_y^2 u = & \triangle u & \text{(parabolic)} & (u(t,x,y) \in R, t \in R, (x,y) \in \mathbb{R}^2) \end{cases}$$

PDE of second order have a long tradition and we recall the Laplace equation (Pierre Simon Laplace 1749-1827)

$$\partial_x^2 u + \partial_y^2 u + \partial_z^2 u = \triangle u = 0 \text{ (elliptic, linear, homogeneous)} \tag{3.1}$$

and its nonhomogeneous version

$$\triangle u = \partial_x^2 u + \partial_y^2 u + \partial_t^2 u = f(x, y, z) \text{ (Poisson equation)} \tag{3.2}$$

originate in the Newton universal attraction law (Isaac Newton 1642-1727). Intuitively, it can be explained as follows. An attractive body induces a field of attraction

where intensity at each point $(x, y, z) \in R^3$ is calculated using Newton's formula

$$u = \gamma \frac{\mu}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}}$$

where γ is a constant, $\mu =$ mass of the body, considering that the attractive body is reduced to the point $(x_0, y_0, z_0) \in R^3$. In the case of several attractive bodies which are placed at the points $(x_i,\, y_i,\, z_i),\ i \in \{1, \ldots, N\}$, we compute the corresponding potential function

$$u = \gamma \sum_{i=1}^{N} \frac{\mu_i}{\sqrt{(x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2}} = \gamma \sum_{i=1}^{N} \frac{\mu_i}{r(P, P_i)}$$

where

$$P = (x, y, z), P_i = (x_i, y_i, z_i)$$

and

$$r(P, P_i) = \sqrt{(x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2}.$$

It was Laplace who proposed to study the corresponding PDE satisfied by the potential function u and in this respect denote $u_i = \gamma \frac{\mu_i}{r(P,P_i)}$ and compute its partial derivatives. We notice that $\partial_x r = \frac{x-x_i}{r}$, $\partial_y r = \frac{y-y_i}{r} = \partial_z r = \frac{z-z_i}{r}$ and

$$\partial_x u_i = -\gamma \mu_i \frac{(x-x_i)}{r^3},\ \partial_y u_i = -\gamma \mu_i \frac{(y-y_i)}{r^3},\ \partial_z u_i = -\gamma \mu_i \frac{(z-z_i)}{r^3} \tag{3.3}$$

Using (3.3) we see easily that

$$\begin{cases} \partial_x^2 u_i = \gamma \mu_i [-\frac{1}{r^3} + 3\frac{x-x_i{}^2}{r^5}] \\ \partial_y^2 u_i = \gamma \mu_i [-\frac{1}{r^3} + 3\frac{y-y_i{}^2}{r^5}] \\ \partial_z^2 u_i = \gamma \mu_i [-\frac{1}{r^3} + 3\frac{z-z_i{}^2}{r^5}] \end{cases} \tag{3.4}$$

and by adding we obtain

$$\triangle u_i = \partial_x^2 u_i + \partial_y^2 u_i + \partial_z^2 u_i = 0, \quad i = 1, 2, \ldots, N \tag{3.5}$$

which implies

$$(u = \sum_{i=1}^{N} u_i) \triangle u = \partial_x^2 u + \partial_y^2 u + \partial_z^2 u = 0 \ \textbf{(Laplace Equation)} \tag{3.6}$$

3.2 Poisson Equation

It may occur that we need to consider a body with a mass distributed in a volume having the density $\rho = \rho(a, b, c)$ at the point $x = a, y = b, z = c$ and vanishing outside

of the ball $a^2+b^2+\mathcal{C}^2 \leqslant R^2$. In this case the potential function will be computed as follows

$$u(P) = \iiint\limits_{a^2+b^2+\mathcal{C}^2\leqslant R^2} \frac{\rho(a,b,c)\,da\,db\,dc}{r(P,P(a,b,c))} \tag{3.7}$$

where $P=(x,y,z)$, $P(a,b,c)=(a,b,c)$ and the constant γ is included in the function ρ. By a direct computation we will prove that if $\rho(a,b,c)$, $(a,b,c)\in B(0,R)$, is first order continuously differentiable then the potential defined in (3.7) satisfies Poisson equation.

$$\triangle u \;=\; \partial_x^2 u + \partial_y^2 u + \partial_z^2 u \;=\; -4\pi\rho(x,y,z),\; (x,y,z)\in B(0,R) \tag{3.8}$$

and

$$\triangle u \;=\; 0,\;\; (x,y,z)\notin B(0,R)\;\text{(Laplace Equation)} \tag{3.9}$$

We recall that a similar law of interaction between electrical particles is valid and Coulomb law is described by

$$\mu \;=\; \gamma\frac{m_1 m_2}{r^2},\;\; (m_1,m_2)-\;\text{electric charges of } (P_1,P_2) \tag{3.10}$$

$\gamma \;=\; \frac{1}{\epsilon}$ a constant,

$$r^2 \;=\; (x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2\; P_1=(x_1,y_1,z_1)P_2=(x_2,y_2,z_2).$$

The associated electrostatic field (E_x,E_y,E_z) is defined by

$$E_x \;=\; \partial_x u,\; E_y \;=\; \partial_y u,\; E_z \;=\; \partial_z u$$

and in this case we get the following Poisson equation

$$\partial_x E_x + \partial_y E_y + \partial_z E_z \;=\; \frac{4\,\pi\,\rho}{\epsilon} \tag{3.11}$$

when a density ρ is used and the electrical potential function has the corresponding integral form.

Proof of the equation (3.8) for u defined in (3.7). Rewrite the potential function on the whole space

$$u(x,y,z) \;=\; \iiint \frac{\rho(a,b,c)}{\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}}\,da\,db\,dc \tag{3.12}$$

and making a translation of coordinates $a-x \;=\; \xi$, $b-y \;=\; \eta$, $c-z \;=\; \tau$, we get

$$u(x,y,z) \;=\; \iiint \frac{\rho(x+\xi,\, y+\eta,\, z+\tau)}{\sqrt{\xi^2+\eta^2+\tau^2}}\,d\xi\,d\eta\,d\tau \tag{3.13}$$

where the integral is singular and $\xi=\eta=\tau=0$ is the singular point. The integral in (3.13) is uniformly convergent with respect to the parameters (x,y,z) because the

function under integral has an integrable upper bound $\frac{\rho^*}{\sqrt{\xi^2+\eta^2+\tau^2}}$, where $\rho^* = \max|\rho|$. In this respect, we notice that if $(x, y, z) \in B(0, R) \subseteq R^3$ then $B(0, 2R)$ can be taken as a domain D where the integration of (3.13) is performed. (see $\xi^2+\eta^2+\tau^2 \leqslant (2R)^2$). Taking a standard coordinates transformation (spherical coordinates)

$$\xi = r\cos\varphi\sin\psi,\ \eta = r\sin\varphi\sin\psi,\ \sigma = r\cos\psi,\ 0 \leqslant \varphi \leqslant 2\pi,\ 0 \leqslant \psi \leqslant \pi,\ 0 \leqslant r \leqslant 2R$$

we rewrite (3.13) on $D = B(0, 2R)$ as follows

$$\iiint\limits_D \frac{\rho^* d\xi d\eta d\sigma}{\sqrt{\xi^2+\eta^2+\sigma^2}} = 4\pi\rho^* \int_0^{2R} \frac{r^2 dr}{r} = 4\pi\rho^* \frac{(2R)^2}{2} \tag{3.14}$$

where

$$\iint\limits_D \frac{1}{r} d\xi d\eta d\sigma = (\int_0^{2R} \frac{r^2}{r} dr(\int_0^{\pi} sin\psi d\psi).2\pi = 4\pi \int_0^{2R} rdr$$

and

$$det \begin{pmatrix} \partial_\nu \xi \\ \partial_\nu \eta \\ \partial_\nu \sigma \end{pmatrix} = -r^2 sin\psi(\nu = (r, \varphi, \psi)) \text{ are used .}$$

In addition by formal derivation of (3.13) with respect to(x, y, z) we get the following uniformly convergent integrals

$$\begin{aligned} \partial_x u &= \iiint \frac{\partial_x[\rho(x+\xi, y+\ eta, z+\sigma)]}{\sqrt{\xi^2+\eta^2+\sigma^2}} d\xi d\eta d\sigma \\ &== \iiint\limits_D \frac{\partial_\xi[\rho(x+\xi, y+\ eta, z+\sigma)]}{\sqrt{\xi^2+\eta^2+\sigma^2}} d\xi d\eta d\sigma \end{aligned} \tag{3.15}$$

Using $\rho(x+\xi, y+\eta, z+\sigma) = 0$ on the sphere $\xi^2+\eta^2+\sigma^2 = (2R)^2$ and

$$\frac{\partial_\xi[\rho(x+\xi, y+\eta, z+\sigma)]}{\sqrt{\xi^2+\eta^2+\sigma^2}} = \partial_\xi[(\rho(x+\xi, y+\eta, z+\sigma))]\sqrt{\xi^2+\eta^2+\sigma^2} + \frac{\xi\rho(x+\xi+\eta, z+\sigma)}{(\xi^2+\eta^2+\sigma^2)^{3\backslash 2}}$$

we get

$$\partial_x u = \iiint_d \frac{\xi\rho(x+\xi+\eta, z+\sigma)}{(\xi^2+\eta^2+\sigma^2)^{3\backslash 2}} d\xi d\eta \tag{3.16}$$

provided

$$0 = \iint\limits_D [\xi\phi(\xi, \eta, \sigma)] d\xi d\eta d\sigma = \iint\limits_{D1} [\phi(\xi_2, \eta_1, \sigma) - \phi(\xi_1, \eta_1, \sigma)] d\eta d\sigma$$

is used where

$$\xi_2 = +\sqrt{(2R^2) - \eta^2 - \sigma^2},\ \xi_1 = -\sqrt{(2R^2) - \eta^2 - \sigma^2}$$

and $\phi(\xi_i, \eta, \sigma) = 0\ i \in \{1, 2\}$. The integral (3.16)is uniformly convergent with respect to$(x, y, z) \in D(0, R)$ and noticing $\frac{\xi}{\sqrt{\xi^2+\eta^2=\sigma^2}} \leqslant 1$, we get the following integrable

upper bound

$$|\partial_x u| \leqslant \iiint\limits_D \frac{\rho^* d\xi d\eta d\sigma}{\sqrt{\xi^2+\eta^2+\sigma^2}} = 4\pi\rho^* \int_0^{2R} \frac{r^2 dr}{r^2} = 4\pi\rho^2(2R) \tag{3.17}$$

which proves that (3.16) is valid. Similar arguments are used to show that $\partial_y u$ and $\partial_z u$ exist fulfilling

$$\begin{cases} \partial_y u = \iint\limits_D \frac{\eta\rho(x+\xi_1 y+\eta_1 z+\sigma)}{(\xi^2+\eta^2+\sigma^2)^{3\backslash 2}} d\xi d\eta d\sigma \\ \partial_z u = \iint\limits_D \frac{\sigma\rho(x+\xi_1 y+\eta_1 z+\sigma)}{(\xi^2+\eta^2+\sigma^2)^{3\backslash 2}} d\xi d\eta d\sigma \end{cases} \tag{3.18}$$

Applying ∂_x to $(\partial_x u)$ in (3.16),∂_y to $\partial_y u$ and ∂_z to $(\partial_z u)$ in (3.18) we get convergent integrals

$$\begin{cases} \partial_x^2 u = \iiint\limits_D \frac{\xi}{(\xi^2+\eta^2+\sigma^2)^{3\backslash 2}} \partial_\xi[\rho(x,\xi,y,\eta,z+\sigma)] d\xi d\eta d\sigma \\ \partial_y^2 u = \iiint\limits_D \frac{\eta}{(\xi^2+\eta^2+\sigma^2)^{3\backslash 2}} \partial_\eta[\rho(x,\xi,y,\eta,z+\sigma)] d\xi d\eta d\sigma \\ \partial_z^2 u = \iiint\limits_D \frac{\sigma}{(\xi^2+\eta^2+\sigma^2)^{3\backslash 2}} \partial_\sigma[\rho(x,\xi,y,\eta,z+\sigma)] d\xi d\eta d\sigma \end{cases} \tag{3.19}$$

where $\partial_s[\rho(x+\xi,y+\eta,z+\sigma)](s \in \xi,\eta,\sigma)$ is a continuous and bounded function on D(see ρ is first order continuously differentiable on D) using (3.19) we get the expression of the laplacian

$$\begin{cases} \Delta u = \partial_x^2 u + \partial_y^2 u + \partial_z^2 u \\ = \iiint\limits_D \frac{\xi\partial_\xi+\eta\partial_\eta+\sigma\partial_\sigma[\rho(x+\xi,y+\eta,z+\sigma)]}{(\xi^2+\eta^2+\sigma^2)^{3}2} d\xi d\eta d\sigma \\ = \int_0^{2R}\{\iint\limits_{S_r} \frac{\partial_r[\rho(x+\xi,y+\eta,z+\sigma)]}{r^2} dS_r\} dr \end{cases} \tag{3.20}$$

where S_r is a sphere with radius

$$r = (\xi^2+\eta^2+\sigma^2)^{\frac{1}{2}}$$

and

$$\partial_r\rho = < (\partial_\xi\rho, \partial_\eta\rho, \partial_\sigma\rho), (\frac{\xi}{r}, \frac{\eta}{r}, \frac{\sigma}{r}) >$$

Here each $(\xi,\eta,\sigma) \in S_r$ can be represented as

$$(\xi,\eta,\sigma) = (r\xi_0, r\eta_0, r\sigma_0)$$

where

$$(\xi_0,\eta_0,\sigma_0) \in S_1$$

and using $d\,S_r = r^2 d\,S_1$ we rewrite (3.20) as follows

$$\Delta u \begin{cases} = \int_0^{2R}\{\iint\limits_{S_1} \partial_r[\rho(x+\xi_0 r, y+\eta_0 r, z+\sigma_0 r)]dr\}dS_1 \\ = \iint\limits_{S_1}[\rho(x+2R\xi_0, y+2R\eta_0, z+2R\sigma_0) - \rho(x,y,z)]dS_1 \\ - \iint\limits_{S_1} \rho(x,y,z)dS_1 = -4\pi\rho(x,y,z) \end{cases} \tag{3.21}$$

It shows that the equality (3.8) (Poisson equation) is for any $(x, y, z) \in B(0, R)$ where $R > 0$ was arbitrarily fixed and the proof is complete. $\square$
We conclude this introduction by showing that the potential function $u(x, y, z)$ defined in (3.7) satisfied the following asymptotic behaviour

$$\lim_{r\to\infty} u(x,y,z) = 0, \text{ or } \lim_{r\to\infty} \sqrt{x^2+y^2+z^2}\,u(x,y,z) = \iint\limits_D\int \rho(a,b,c)da\,db\,dc \tag{3.22}$$

In this respect ,denote $Q = (x, y, z)$, $P = (a, b, c)$, $dv = da\,db\,dc$ and rewrite equation (3.22) as follows

$$u(Q) = \iiint\limits_D \frac{\rho(P)dv}{r(P,Q)} \quad (D : r(P,Q) \leqslant R)$$

$$\lim_{r(0,Q)\to\infty} r(0,Q)u(Q) = \lim_{r(0,Q)\to\infty} \iiint\limits_D \frac{\rho(P)dv}{[1 - \frac{r(0,Q)-r(P,Q)}{r(0,Q)}]} = \iiint\limits_D \rho(P)dv \tag{3.23}$$

Here we notice that $a = \frac{r(0,Q)-r(P,Q)}{r(0,Q)}$ satisfies $\mid a \mid < 1\,,\frac{1}{(}1-a) = 1 + a + a^2 +, \ldots$ and $\mid r(0,Q) - r(P,Q) \mid \leqslant r(0,P)$ leads us to (3.23) and (3.22) are valid. We conclude the above given considerations by

Theorem 3.2.1. *Let $\rho(x, y, z) : \mathbb{R}^3 \to \mathbb{R}$ be a continuously differentiable function in a open neighborhood $V(x_0, y_0, z_0) \subseteq \mathbb{R}^3$ and vanishing outside of the fixed ball $B(0, L) \subseteq \mathbb{R}^3$. Then*

$$u(x,y,z) = \iiint\limits_{a^2+b^2+\mathcal{C}^2\leqslant L^2} \frac{\rho(a,b,c)da\,db\,dc}{r(P,P(a,b,c))}\ , P = (x,y,z) \in \mathbb{R}^3$$

satisfies the following Poisson equations

$$\Delta u(x,y,z) = (\partial_x^2 + \partial_y^2 + \partial_z^2)u(x,y,z) = -4\pi\rho(x,y,z) \tag{3.24}$$

for any $(x, y, z) \in V(x_0, y_0, z_0), \Delta u(x, y, z) = 0\ \forall\,(x, y, z)$ not in $B(0, L)$. In addition

$$\lim_{r\to\infty} u(x,y,z) = 0\,(or \lim_{r\to\infty} ru(x,y,z)) = \iiint\limits_{B(0,L)} \rho(a,b,c)da\,db\,dc \tag{3.25}$$

Remark 3.2.2. *The result in Theorem 3.2.1 holds true when replacing $\mathbb{R}^3$ with $\mathbb{R}^n (n \geqslant 3)$ and if it is the case then the corresponding newtonian potential function*

is given by

$$u(x) = \underbrace{\int \cdots \int}_{n\,times} \frac{\rho(y)dy_1 \ldots dy_n}{r^{n-2}}$$

where $\rho(y) : \mathbb{R}^n \to \mathbb{R}$ *is a continuously differentiable function vanishing outside of the ball* $B(0, L) \subseteq \mathbb{R}^n$ *and* $r = \mid x - y \mid = (\sum_{i=1}^{n}(x_i - y_i)^2)^{\frac{1}{2}}$. *The corresponding Poisson equation is given by*

$$\sum_{i=1}^{n} \partial_i^2 u(x) = \Delta u(x) = \rho(x)(-\sigma_1) \ \forall\, x \in \mathbb{R}^n, \ \textit{where } \sigma_1 = meas S(0,1)$$

3.3 Exercises

Using the same algorithm as in Theorem 3.2.1 prove that the corresponding potential function(logarithm)in $\mathbb{R}^2$ is given by

$$u(x,y) = \iint \rho(a,b)\{ln \frac{1}{((x-a)^2 + (y-b)^2)^{\frac{1}{2}}}\} da\, db$$

and satisfies the following Poisson equation

$$\Delta u(x,y) = (\partial_x^2 + \partial_y^2)u(x,y) = -2\pi\rho(x,y), \ \forall\,(x,y) \in V(x_0, y_0)$$

Here $\rho : \mathbb{R}^2 \to \mathbb{R}$ is first order continuously differentiable in the open neighborhood $V(x_0, y_0) \subseteq \mathbb{R}^2$ and vanishes out side of the disk $B(0, L) \subseteq \mathbb{R}^2$. By a direct computation show that the following Lapace equation

$$\Delta u(x) = \sum_{i=1}^{n} \partial_i^2 u(x) = 0 \ \ \forall\, x \in \mathbb{R}^n, \ (n \geqslant 3), \ x \neq 0$$

is valid, where

$$u(x) = \frac{1}{r^{n-2}}, r = (\sum_{i=1}^{n} x_i^2)^{\frac{1}{2}}, \ x = (x_1, ...x_n)\partial_i^2 u(x) = \frac{\partial^2 u(x)}{\partial x_i^2}$$

3.4 Maximum Principle for Harmonic Functions

Any solution of the Lapace equation $\Delta u(x,y,z) = 0$ will be called harmonic function
Maximum principle
A harmonic function $u(x,y,z)$ which is continuous in a bounded closed domain $\overline{G} =$

$G \sqcup \Gamma$ and admitting second order continuous partial derivatives in the open set $G \subseteq \mathbb{R}^3$ satisfy

$$\max_{(x,y,z)\in\overline{G}} u(x,y,z) = \max_{(x,y,z)\in\Gamma} u(x,y,z) \, and \, \min_{(x,y,z)\in\overline{G}} u(x,y,z) = \min_{(x,y,z)\in\Gamma} u(x,y,z)$$

where $\Gamma = \partial G$ (boundary of G)

Proof. Denote $m = \max\{u(x,y,z) : (x,y,z) \in \Gamma\}$ and assume that $\max\{u(x,y,z) : (x,y,z) \in \overline{G}\} = M = u(x_0,y_0,z_0) > m$, where $(x_0,y_0,z_0) \in G$. Define the auxiliary function

$$\nu(x,y,z) = u(x,y,z) + \frac{M-m}{2d^2}[(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2]$$

and

$$d = \max\{r(P,Q) : P,Q \in \overline{G}\} \text{ and } r(P,Q) =\mid P-Q \mid$$

(distance between two points). By definition,

$$r^2(P,P_0) = (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2, (P = (x,y,z), P_0 = (x_0,y_0,z_0))$$

and using $r^2(P,P_0) \leqslant d^2$
we get

$$\nu(x,y,z) \leqslant m + \frac{M-m}{2} = \frac{M+m}{2}, \forall\, (x,y,z) \in \Gamma$$

On the other hand, $\nu(x_0,y_0,z_0) = u(x_0,y_0,z_0) = M$ and it implies $\max\{\nu(x,y,z) : (x,y,z) \in \overline{G}\} = V(\overline{P})$ is achieved in the open set $\overline{P} \in G$. As a consequence

$$\partial_x \nu(\overline{x},\overline{y},\overline{z}) = 0,\ \partial_y \nu(\overline{x},\overline{y},\overline{z}) = 0,\ \partial_z \nu(\overline{x},\overline{y},\overline{z}),\ \partial_x^2 \nu(\overline{x},\overline{y},\overline{z}) \leqslant 0$$

$$\partial_y^2 \nu(\overline{x},\overline{y},\overline{z}) \leqslant 0 \text{ and } \partial_z^2 \nu(\overline{x},\overline{y},\overline{z}) \leqslant 0$$

where $\overline{P} = (\overline{x},\overline{y},\overline{z})$. In addition

$$\Delta\nu(\overline{P}) \leqslant 0 \text{ and } \Delta\nu(\overline{P}) = \Delta u(\overline{P}) + \frac{M-m}{2d^2}[\Delta r^2(P,P_0)]_{P=\overline{P}} = \frac{M-m}{2d^2}(2+2+2) > 0$$

which is a contradiction. Therefore

$$u(x,y,z) \leqslant m = \max\{u(x,y,z) : (x,y,z) \in \Gamma\}, \forall\, (x,y,z) \in \overline{G}$$

To prove the inequality

$$u(x,y,z) \geqslant \min\{u(x,y,z) : (x,y,z) \in \Gamma\}, \forall\, (x,y,z) \in \overline{G}$$

we apply the above given result to $\{-u(x,y,z)\}$. The proof is complete. □

Remark 3.4.1. *With the same proof we get that a harmonic function in the plane*

$$\partial_x^2 u(x,y) + \partial_y^2 u(x,y) = 0 (x,y) \in D \subseteq \mathbb{R}^2$$

which is continuous on

$$\overline{D} = D U \partial D \quad satisfy$$

$$max\{u(x,y) : (x,y) \in \overline{D}\} = max\{u(x,y) : (x,y) \in \partial D\}$$
$$min\{u(x,y) : (x,y) \in \overline{D}\} = min\{u(x,y) : (x,y) \in \partial D\}$$

Remark 3.4.2. *The Poisson equation*

$$\Delta u(x,y,z) = -4\pi\rho(x,y,z) \tag{3.26}$$

has a unique solution under the restriction.

$$\lim_{r\to\infty} u(x,y,z) = 0 \text{ where } \rho(x,y,z) : \mathbb{R}^3 \to \mathbb{R}$$

is a continuous function. Consider that u_1, u_2 are solutions of the Poisson equation (3.26) *fulfilling*

$$\lim_{r\to\infty} u_i(x,y,z) = 0, i \in \{1,2\}$$

Then $u = u_1 - u_2$ satisfies $\Delta u(x,y,z) = 0$ for any $(x,y,z) \in \mathbb{R}^3$ and $\lim_{r\to\infty} u(x,y,z) = 0$. In particular, $\{u(x,y,z) : (x,y,z) \in B(0,R) \subseteq \mathbb{R}^3\}$ satisfies, maximum principle, where $B(0,R) = \overline{G}$ and $\Gamma = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = R^2\}$. We get $u(x_0,y_0,z_0) \leqslant max\{u(x,y,z) : (x,y,z) \in \Gamma\}$ and $u(x_0,y_0,z_0) \geqslant min\{u(x,y,z) : (x,y,z) \in \Gamma\}$ for any $(x_0,y_0,z_0) \in int\, B(0,R)$. In particular, for $(x_0,y_0,z_0) \in int\, B(0,R)$ fixed and passing $R \to \infty$ we get $u(x_0,y_0,z_0) = 0$ and $u_1(x,y,z) = u_2(x,y,z)$ for any $(x,y,z) \in \mathbb{R}^3$.

3.4.1 The Wave Equation; Kirchhoff, D'Alembert and Poisson Formulas

Consider the waves equation

$$\partial_t^2 u(t,x,y,z) = C_0^2(\partial_x^2 u + \partial_y^2 u + \partial_z^2 u)(t,x,y,z) \tag{3.27}$$

for $(x,y,z) \in D(domain) \subseteq R^3$ and initial condition

$$u(0,x,y,z) = u_0(0,x,y,z),\ \partial_t u(0,x,y,z) = u_1(x,y,z) \tag{3.28}$$

The integral representation of the solution satisfying (3.27) and (3.28)is called Kirchhoff formula. In particular, for 1- dimensional case the equation and initial conditions are described by

$$\partial_t^2 u(t,x) = C_0^2 \partial_x^2 u(t,x), u(0,x) = u_0(x), \partial_t u(0,x) = u_1(x) \tag{3.29}$$

and d'Alembert formula gives the following representation

$$u(t,x) = \frac{u_0(x+c_0t) + u_0(x-c_0t)}{2} + \frac{1}{2c_0}\int_{x-c_0t}^{c_0t+x} u_1(\sigma)d\sigma \tag{3.30}$$

In this simplest case, the equation (3.29)is decomposed as follows

$$(\partial_t + c_0\partial_x)(\partial_t - c_0\partial_x)u(t,x) = 0 \tag{3.31}$$

and find the general solution of the linear first order equation

$$\partial_t\nu(t,x) + c_0\partial_x\nu(t,x) = 0 \tag{3.32}$$

By a direct computation we get

$$\nu(t,x) = \nu_0(x - c_0t),\ (t,x) \in \mathbb{R}\times\mathbb{R} \tag{3.33}$$

where $\nu_0(\lambda) : \mathbb{R} \to \mathbb{R}$ is an arbitrary first order continuously differentiable function. Then solve the following equation

$$\begin{cases} \partial_t u(t,x) - c_0\partial_x u(t,x) = \nu_0(x - c_0t) \\ u(0,x) = u_0(x),\ \partial_t u(0,x) = u_1(x) \end{cases} \tag{3.34}$$

From the equation $u(0,x) = u_0(x)$, $\partial_t u(0,x) = u_1(x)$ we find ν_0 such that

$$u_1(x) - c_0\partial_x u_0(x) = \nu_0(x),\ x \in \mathbb{R} \tag{3.35}$$

and using the characteristic system associated with (3.34) we obtain

$$u(t,x) = u_0(x + c_0t) + \int_0^t \nu_0(x + c_0t - 2c_0s)ds \tag{3.36}$$

Using (3.35) into (3.34) we get the corresponding D'Alembert formula

$$u(t,x) = \frac{u_0(x + x_0t) + u_0(x - c_0t)}{2} + \frac{1}{2c_0}\int_{x-c_0t}^{x+c_0t} u_1(\sigma)d\sigma \tag{3.37}$$

given in (3.30) In the two-dimensional case $(x,y) \in \mathbb{R}^2$ we recall that the Poisson formula is expressed as follows

$$\begin{aligned} u(t,x,y) &= \frac{1}{2\pi c_0}[\partial_t(\int_0^{2\pi}\int_0^{c_0t} \rho\frac{u_0(x + \rho cos\varphi, y + \rho sin\varphi)}{\sqrt{c_0^2t^2 - \rho^2}}d\rho d\varphi) \\ &+ (\int_0^{2\pi}\int_0^{c_0t} \rho\frac{u_1(x + \rho cos\varphi, y + \rho sin\varphi)}{\sqrt{c_0^2t^2 - \rho^2}}d\rho d\varphi)] \end{aligned} \tag{3.38}$$

The general case ,$(x,y,z) \in \mathbb{R}^3$, will be treated reducing the equation (3.27) to an wave equation analyzed in the one-dimensional case provided adequate coordinate transformations are used. In this respect, for a $P_0 = (x_0, y_0, z_0) \in D$ fixed and

$B(P_0, r) \subseteq D$ we define ($\overline{u}$ is the mean value of u on S_r)

$$\overline{u}(r,t) = \frac{1}{4\pi r^2} \iint_{S_r} u(t,x,y,z)dS_r = \frac{1}{4\pi} \iint_{S_1} u(t,x,y,z)dS_1 \tag{3.39}$$

where $S_r = \partial B(P_0, r)$ is the boundary of the ball $B(P_0, r)$, and $dS_r = r^2 dS_1$ is used.The integral in (3.39) can be computed as a two dimensional integral provided we notice that each $(x, y, z) \in S_r$ can be written as

$$(x,y,z) = (x_0 + \alpha r, y_0 + \beta r, z_0 + \gamma r) = P_0 + rw \tag{3.40}$$

where $(\alpha, \beta, \gamma) = \omega$ are the following

$$\begin{cases} \alpha = sin\theta cos\varphi \ \ 0 \leqslant \theta \leqslant \pi \\ \beta = sin\theta sin\varphi \ \ 0 \leqslant \varphi \leqslant \pi \\ \gamma = cos\theta \end{cases} \tag{3.41}$$

Here $dS_r = r^2 sin\theta d\varphi d\theta$ and rewrite (3.39) using a two dimensional integral

$$\overline{u}(r,t) = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} u(t, P_0 + rw) sin\theta d\varphi d\theta \tag{3.42}$$

assuming $w = (\alpha, \beta, \gamma)$ given in (3.42).The explicit expression of $\overline{u}(r,t)$ in (14) will be deduced taking into consideration the corresponding wave equation satisfied by $\overline{u}(r,t)$ when $u(t,x,y,z)$ is a solution of (3.27). In this respect, integrate in both sides of (3.27) using the three dimensional domain $D_r = B(P_0, r), P_0 = (x_0, y_0, z_0)$, and the spherical transformation of the coordinates (x, y, z)

$$(x,y,z) = (x_0 + \rho\alpha, y_0 + \rho\beta + z_0 + \rho\gamma) = P_0 + \rho\omega \tag{3.43}$$

where $0 \leqslant \rho \leqslant r$ and $\alpha(\theta, \varphi), \beta(\theta, \varphi), \gamma(\theta)$ satisfy (3.41). We get

$$\iiint_{D_r} \partial_t^2 u(t,x,y,z)dxdydz = \int_0^r (\iint_{S_1} \partial_t^2 u(t, P_0 + \rho\omega)\rho^2 dS_1)d\rho \tag{3.44}$$

where $dS_1 = sin\theta d\theta d\varphi$ and $S_1 = S(P_0, 1)$ is the sphere centered at P_0. Denote $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$ and for the integral in the right hand side we apply Gauss-Ostrogradsky

formula. We get

$$\begin{aligned}C_0^2\iiint\limits_{D_r}\Delta u(t,x,y,z)dxdydz &= C_0^2\iint\limits_{S_r}[\alpha\partial_x u(t,P_0+r\omega)\\ &+ \beta\partial_y u(t,P_0+r\omega)+\gamma\partial_z u(t,P_0+r\omega)]dS_r\\ &= C_0^2\iint\limits_{S_1}\{\partial_r[u(t,P_0+r\omega)]r^2dS_1\}\\ &= C_0^2r^2\partial_r[\iint\limits_{S_1}u(t,P_0+r\omega)dS_1] \qquad (3.45)\end{aligned}$$

where $(\alpha,\beta,\gamma)=\omega$ are the coordinates of the unit outside normal vector at S_r. Using (3.39) we rewrite (3.45) as

$$C_0^2\iiint\limits_{D_r}\Delta u(t,x,y,z)dxdydz = C_0^2r^2(4\pi)\partial_r\overline{u}(r,t) \qquad (3.46)$$

In addition, notice that (3.44) can be written as

$$\begin{aligned}\iiint\limits_{D_r}\partial_t^2u(t,P_0+\rho\omega)dxdydz &= \int_0^r\rho^2[\partial_t^2\iiint\limits_{S_1}u(t,P_0+\rho\omega)dS_1]d\rho\\ &= \int_0^r 4\pi\rho^2\partial_t^2\overline{u}(\rho,t)d\rho \qquad (3.47)\end{aligned}$$

and deriving with respect to r in (3.46)and (3.47) we obtain

$$\partial_t^2\overline{u}(r,t)=\frac{C_0^2}{r^2}\partial_r[r^2\partial_r\overline{u}(r,t)] \qquad (3.48)$$

Denote $\overline{\overline{u}}(r,t)=r\overline{u}(r,t)$ and using (3.48) compute

$$\partial_t^2\overline{\overline{u}}(r,t)\begin{cases} = & r\partial_t^2\overline{u}(r,t)\\ = & \frac{C_0^2}{r}\partial_r[r^2\partial_r\overline{u}(r,t)]\\ = & \frac{C_0^2}{r}[2r\partial_r\overline{u}(r,t)+r^2\partial_r^2\overline{u}(r,t)]\\ = & C_0^2[2\partial_r\overline{u}(r,t)+r\partial_r^2\overline{u}(r,t)]\end{cases} \qquad (3.49)$$

Notice that

$$\partial_r^2\overline{\overline{u}}(r,t)=2\partial_r\overline{u}(r,t)+r\partial_r^2\overline{u}(r,t) \qquad (3.50)$$

and rewrite (3.49) as follows

$$\partial_t^2\overline{\overline{u}}(r,t)=C_0^2\partial_r^2\overline{\overline{u}}(r,t),\ r\geqslant 0 \qquad (3.51)$$

where $\overline{\overline{u}}$ satisfies the boundary condition

$$\overline{\overline{u}}(0,t)=0 \qquad (3.52)$$

Extend $\overline{\overline{u}}(r,t)$ for $r < 0$ by $\overline{\overline{u}}(-r,t) = -\overline{\overline{u}}(r,t)$ where $r \geqslant 0$. In addition, the initial conditions for $\overline{\overline{u}}(r,t)$ are the following

$$\begin{cases} \overline{\overline{u}}(r,0) = r\overline{u}(r,0) = \frac{1}{4\pi r}\iint\limits_{S_r} u_0(x,y,z)dS_r \\ \partial_t\overline{\overline{u}}(r,0) = \frac{1}{4\pi r}\iint\limits_{S_r} u_1(x,y,z)dS_r \end{cases} \tag{3.53}$$

The solution $\{\overline{\overline{u}}(r,t)\}$ fulfilling (3.51))and (3.53) are expressed using Poisson formula (see(3.30))

$$\overline{\overline{u}}(r,t) = \frac{\varphi(r+C_0t)+\varphi(r-C_0t)}{2} + \frac{1}{2C_0}\int_{x-C_0}^{x+C_0}\psi(\sigma)d\sigma \tag{3.54}$$

where

$$\varphi(\xi) = \overline{\overline{u}}(\xi,0) = \frac{1}{4\pi\xi}\iint\limits_{S_\xi} u_1(x,y,z)dS_\xi \tag{3.55}$$

and

$$\psi(\xi) = \partial_t\overline{\overline{u}}(\xi,0) = \frac{1}{4\pi\xi}\iint\limits_{S_\xi} u_0(x,y,z)dS_\xi \tag{3.56}$$

Using (3.55) and (3.56) we get the Kirchhoff formula for the solution of the wave equation (3.27) satisfying Cauchy conditions (3.28) and it can be expressed as follows

$$\begin{aligned} u(t,P_0) &= \frac{d}{dr}[\frac{1}{4\pi r}\iint\limits_{S_r(P_0)} u_0(x,y,z)dS_r]_{r=C_0t} + [\frac{1}{4\pi C_0}\iint\limits_{S_r(P_0)} u_1(x,y,z)dS_r]_{r=C_0t} \\ &= \frac{1}{4\pi C_0}[\frac{d}{dr}(\frac{1}{C_0t}\iint\limits_{S_{c_0t}(P_0)} u_0(x,y,z)dS) + \frac{1}{C_0t}\iint\limits_{S_{c_0t}(P_0)} u_1(x,y,z)dS] \end{aligned} \tag{3.57}$$

where

$$\iint\limits_{S_{c_0t}(P_0)} u_0(x,y,z)dS = \int_0^{2\pi}\int_0^{\pi} u(P_0+(C_0t)\omega)(C_0t)^2 sin\theta d\theta d\varphi$$

and $\omega = (\alpha,\beta,\gamma)$ is defined in (3.41). Passing $P_0 \to P = (x,y,z)$ in the Kirchhoff formula (3.57) we get the integral representation of a Cauchy problem solution satisfying (3.27) and (3.28)

$$u(t,x,y,z) = \frac{1}{4\pi C_0}[\frac{d}{dt}(\frac{1}{C_0t}\iint\limits_{S_{c_0t}(P_0)} u_0(x,y,z)dS) + \frac{1}{C_0t}\iint\limits_{S_{c_0t}(P_0)} u_1(x,y,z)dS] \tag{3.58}$$

where the integral in (3.58) is computed as follows

$$\frac{1}{C_0 t}\iint\limits_{S_{c_0 t}(P_0)} u(\xi,\eta,\sigma)dS = C_0 t\int_0^{2\pi}\int_0^{\pi} u(x+(C_0t)\alpha, y+(C_0t)\beta, z+(C_0t)\gamma) sin\theta d\theta d\varphi \quad (3.59)$$

and (α,β,γ) are defined in (3.41) $\begin{cases} \alpha = sin\theta cos\varphi & 0 \leqslant \theta \leqslant \pi \\ \beta = sin\theta sin\varphi & 0 \leqslant \varphi \leqslant 2\pi \\ \gamma = cos\theta \end{cases}$

Proposition 3.4.3. *Assume that $u_0, u_1 \in \mathcal{C}^4(\mathbb{R}^3)$ are given. Then $\{u(t,x,y,z) : t \in \mathbb{R}_+, (x,y,z) \in \mathbb{R}^3\}$ defined in* (3.58) *is a solution of the hyperbolic equation* (3.27) *satisfying the Cauchy condition* (3.28).

Proof. Notice that if $u(t,x,y,z)$ satisfies the wave equation (3.27) then $\nu(t,x,y,z) = \partial_t u(t,x,y,z)$ verifies also the wave equation (3.27). It allows us to get the conclusion and it is enough to prove that

$$u(t,x,y,z) = C_0 t\int_0^{2\pi}\int_0^{\pi} u(x+(C_0t)\alpha, y+(C_0t)\beta, z+(C_0t)\gamma) sin\theta d\theta d\varphi \quad (3.60)$$

fulfils the wave equation (3.27).In this respect, using (3.60), compute the corresponding derivatives involved in equation (3.27) and we get

$$\begin{aligned} \partial_t u &= \frac{u}{t} + (C_0 t)\partial_t \int_0^{2\pi}\int_0^{\pi} u(x+(C_0t)\alpha, y+(C_0t)\beta, z+(C_0t)\gamma) sin\theta d\theta d\varphi \\ &= \frac{u}{t} + \frac{1}{t}\int_0^{2\pi}\int_0^{\pi} (\partial_\xi u)\alpha + (\partial_\eta u)\beta + (\partial_\sigma)\gamma C_0^2 t^2 sin\theta d\theta d\varphi \qquad (3.61) \end{aligned}$$

Notice that $(\alpha,\beta,\gamma) = n$ represent the unit vector oriented outside of the sphere S_{C_0} and $dS = (C_0 t)^2 sin\theta\, d\theta\, d\varphi$ we rewrite (3.61)

$$\partial_t u = \frac{u}{t} + \frac{I}{t} \quad (3.62)$$

where

$$\begin{aligned} I &= \int_0^{2\pi}\int_0^{\pi} (\partial_\xi u)\alpha + (\partial_\eta u)\beta + (\partial_\sigma)\gamma C_0^2 t^2 sin\theta d\theta d\varphi \\ &= \iint\limits_{S_{C_0 t}} \frac{\partial u}{\partial n} dS \\ &= \iiint\limits_{B(x,y,z;C_0t)} (\partial_\xi^2 u + \partial_\eta^2 u + \partial_\sigma^2 u) d\xi d\eta d\sigma \qquad (3.63) \end{aligned}$$

is written using Gauss-Ostrogrodsky formula,and $B(x,y,z;C_0t)$ is the ball in $\mathbb{R}^3$

centered at $P = (x, y, z)$. From (3.62), deriving again we get

$$\partial_t^2 = \frac{1}{t}\partial_t u - \frac{1}{t^2}u + \frac{1}{t}\partial_t I - \frac{1}{t^2}I = \frac{1}{t}(\frac{u}{t} + \frac{I}{t}) - \frac{u}{t^2} - \frac{I}{t^2} + \frac{1}{t}\partial_t I = \frac{1}{t}\partial_t I \tag{3.64}$$

On the other hand,using (3.63) and applying the spherical coordinate transformation in the three-dimensional integral we get

$$\begin{cases} I = \int_0^{C_0 t}[\int_0^{2\pi}\int_0^{\pi}(\partial_\xi^2 u + \partial_\eta^2 u + \partial_\sigma^2 u) sin\theta d\theta d\varphi] r^2 dr \\ \partial_t I = C_0 \iint\limits_{S_{C_0}t} (x, y, z)(\partial_\xi^2 u + \partial_\eta^2 u + \partial_\sigma^2 u) ds \end{cases} \tag{3.65}$$

and (3.64) becomes

$$\partial_t^2 u = \frac{C_0}{t} \iint\limits_{S_{C_)}t(x,y,z)} (\Delta u)(\xi, \eta, \sigma) dS \tag{3.66}$$

Denote $\Delta u(t, x, y, z) = (\partial_x^2 u + \partial_y^2 u + \partial_z^2 u)(t, x, y, z)$ and a direct computation applied in (3.60) allows to get

$$\Delta u(t, x, y, z) = \frac{1}{(C_0)^2} . \frac{C_0}{t} \iint\limits_{S_{C_)}t(x,y,z)} (\Delta u)(\xi, \eta, \sigma) dS \tag{3.67}$$

and each term entering in the definition (3.58) will satisfy the wave equation (3.27). In addition, using (3.59), we see easily that the Cauchy condition given in (3.28) are satisfied by the function $u(t, x, y, z)$ defined in (3.58). The proof is complete. □

3.5 Exercises

(a_1) consider the wave equation in plane

$\partial_t^2 u(t, x, y) = c_0^2[\partial_x^2 u(t, x, y) + \partial_y^2 u(t, x, y)], u(0, x, y) = y_0(x, y),\ \partial_t u(0, x, y) = u_1(x, y)$

Find the integral representation of its solution. (Poisson formula see (3.68)).
Hint. It will be deduced from the Kirchhoff formula (see proposition 1) considering that the variable z does not appear. We get

$$\begin{aligned} u(t, x, y) &= \frac{1}{2\pi C_0}\Big[\partial_t\Big(\int_0^{2\pi}\int_0^{C_0 t}\frac{\rho}{\sqrt{c_0^2 t^2 - \rho^2}} u_0(x + \rho cos\phi, y + \rho sin\phi)) d\rho d\phi\Big) \\ &+ \int_0^{2\pi}\int_0^{C_0 t}\frac{\rho}{\sqrt{c_0^2 t^2 - \rho^2}} u_1(x + \rho cos\phi, y + \rho sin\phi) d\rho d\phi\Big] \end{aligned} \tag{3.68}$$

In the Kirchhoff formula (see(3.58)) the function $u(\xi,\eta)$ does not depend on σ and a direct computation gives the following

$$(\theta \in [0,\pi], \phi \in [0,2\pi])$$

$$\begin{aligned} E &= \int_0^{2\pi}\int_0^{\pi} u(x+(C_0t)sin\theta cos\phi, y+(C_0t)sin\theta sin\phi)(C_0^2t^2)sin\theta d\theta d\phi \\ &= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} u()(C_0^2t^2)sin\theta d\theta d\phi + \int_0^{2\pi}\int_{\frac{\pi}{2}}^{\pi} u()C_0^2t^2 sin\theta d\theta d\phi \\ &= 2\int_0^{2\pi}\int_0^{\frac{\pi}{2}} u(x, cotsin\theta cos\phi, y+(cot)sin\theta sin\phi)(C_0^2t^2)sin\theta d\theta d\phi \end{aligned}$$

Change the coordinate $\rho = (C_0t)sin\theta$ and get $d\rho = (C_0t)cos\theta d\theta$ for $0 \leqslant \theta \leqslant \frac{\pi}{2}$ where

$$cos\theta = \sqrt{1-sin^2\theta} = \sqrt{1-\frac{\rho^2}{C_0t}^2} = \frac{1}{C_0t}\sqrt{(C_0t)^2-\rho^2}$$

We get

$$E = 2\int_0^{2\pi}\int_0^{C_0t} \frac{\rho}{\sqrt{c_0^2t^2-\rho^2}} u(x+\rho cos\varphi, y+\rho sin\varphi)d\rho d\phi$$

and Kirchhoff's formula becomes Poison formula given in (3.68).
(a_2) Prove that the solution of the non homogeneous waves equation.

$$\begin{cases} \partial_t^2 u - C_0^2(\partial_x^2 u + \partial_y^2 y + \partial_z^2 u) = f(x,y,z,t) \\ u(0,x,y,z) = 0 \ \ \partial_t u(0,x,y,z) = 0 \end{cases} \tag{3.69}$$

is presented by the following formula.

$$u(t,x,y,z) = \frac{1}{4\pi C_0^2} \iiint\limits_{B(x,y,z;C_0t)} \frac{f(\xi,\eta,\sigma,t-\frac{\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\sigma)^2}}{C_0})}{\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\sigma)^2}} d\xi d\eta d\sigma \tag{3.70}$$

solution.
Denote $p=(x,y,z), q=(\xi,\eta,\sigma)$ and $\mid p-q \mid = [\sum_{i=1}^{3}(p_i-q_i)^2]^{\frac{1}{2}}$; $B(p,C_0t)$ is the ball centered at p with radius C_0t. To get(3.70) as a solution of (3.69) we use the solution of the following homogeneous equation

$$\partial_t^2\nu = c_0^2\Delta\nu, \nu|_{t=\sigma} = \nu_0(p), \partial_t\nu|_{t=\sigma} = f(p,\sigma) = \nu_1(p) \tag{3.71}$$

where the initial moment $t=\sigma(replacing\ t=0)$ is a parameter.Recall the Kirchhoff formula for the solution satisfying (3.71)

$$\nu(t,p) = \frac{1}{4\pi C_0}[\frac{d}{dt}(\iint\limits_{S_r(P)} \frac{\nu_0(q)}{r}dS_r) + \iint\limits_{S_r(P)} \frac{\nu_1(q)}{r}dS_r] \tag{3.72}$$

where $r = C_0 t$ and $dS_r = r^2 dS_1 = r^2 sin\theta d\theta d\varphi$ for$\theta \in [0, 2\pi]$, where S_1 is the unit sphere. Making use of the spherical coordinates transformation

$$q = p + r\omega,\ \omega = (sin\theta cos\varphi, sin\theta sin\varphi, cos\theta),\ r = C_0 t \tag{3.73}$$

we rewrite (3.72) as follows

$$\nu(t,p) = \frac{1}{4\pi}\partial_t[t \iint\limits_{S_1(P)} \nu_0(p + r\omega)dS_1] + \frac{t}{4\pi} \iint\limits_{S_1(P)} \nu_1(p + r\omega)dS_1 \tag{3.74}$$

where $dS_1 = sin\theta\, d\theta\, d\varphi$. Using the Cauchy conditions of (3.71) from (3.74) we get

$$\nu(t,p;\sigma) = \frac{t-\sigma}{4\pi} \iint\limits_{S_1(P)} f(p + C_0(t-\sigma)\omega;\sigma)dS_1,\ 0 \leqslant \sigma \leqslant t \tag{3.75}$$

where $t - \sigma$ representing t and $\nu|_{t=\sigma} = 0$ are used. We shall show that $u(t,p)$ defined by (Duhamel integral)

$$u(t,p) = \int_0^t \nu(t,p;\sigma)\, d\sigma \tag{3.76}$$

is a solution of the wave equation (3.69). The initial conditions $u(0,p) = 0 = \partial_t u(0,p)$ are easily verified by direct inspection. Computing lapacian operator $\Delta_p = \partial_x^2 + \partial_y^2 + \partial_z^2$ applied in both sides of (3.76) we obtain

$$\Delta_p u(t,p) = \int_0^t \Delta_p \nu(t,p;\sigma)\, d\sigma \tag{3.77}$$

Notice that "∂_t" applied in (3.76) lead us to

$$\begin{cases} \partial_t u(t,p) = \nu(t,p;\sigma)_{\sigma=t} + \int_0^t \partial_t \nu(t,p;\sigma)\, d\sigma \\ \partial_t^2 u(t,p) = \partial_t \nu(t,p;\sigma)_{\sigma=t} + \int_0^t \partial_t^2 \nu(t,p;\sigma)\, d\sigma = \\ = f(p,t) + C_0 \int_0^t \Delta_p \nu(t,p;\sigma) d\sigma \\ = f(p,t) + C_0^2 \Delta_p u(t,p) \end{cases} \tag{3.78}$$

and $\{u(t,p)\}$ defined in (3.76) verifies the wave equation (3.69). Using (3.75) we rewrite (3.76) as in conclusion (3.70). In this respect, substituting (3.75) into (3.76) we get following formula ($dS_1 = sin\theta\, d\theta\, d\varphi$)

$$u(t,p;\sigma) = \frac{1}{4\pi} \int_0^t (t-\sigma)[\iint\limits_{S_1(P)} f(p + C_0(t-\sigma)\omega;\sigma)dS_1]d\sigma \tag{3.79}$$

Making a change of variables $C_0(t - \sigma) = r$, we get

$$u(t,p) = \frac{1}{4\pi C_0^2} \int_0^{C_0 t} \int_0^{2\pi} \int_0^{\pi} f(p + r\omega, t - \frac{r}{C_0}) r\, sin\theta\, d\theta\, d\varphi\, dr \tag{3.80}$$

and noticing that $r = \frac{r^2}{r}$, $\mid \omega \mid^2 = 1$ we rewrite (3.80) as follows

$$u(t,p) = \frac{1}{4\pi C_0^2} \iiint\limits_{B(p,C_0 t)} f(q, t - \frac{\mid p-q \mid}{C_0}) \frac{1}{\mid p-q \mid} dq \tag{3.81}$$

and the proof is complete. $\square$

3.5.1 System of Hyperbolic and Elliptic Equations(Definition)

Definition 3.5.1. *A first order system of n equations*

$$A_0 \partial_t u + \sum_{i=1}^{m} A_i(x,t) \partial_i u = f(x,y,u), u \in R^n, \times \in R^m \tag{3.82}$$

is t-hyperbolic if the corresponding characteristic equation det $|| \, \sigma A_0(x,t) + \sum_{i=1}^{m} \xi A_i(x,t) \, || = o$ has n real distinct roots for the variable σ at each point $(t,x) \in [0,T] \times D,\ D \subseteq R^m,$ for any $\xi \in R^m, \xi \neq 0$. There is a particular first order system for which a verification of this property is simple

Definition 3.5.2. *The first order system* (3.82) *is symmetric t- hyperbolic (defined by Friedreich)if all matrices $A_j(x,t) j \in 0,1,...,m$ are symmetric and $A_0(x,t)$ is strictly positive definite in the domain $(t,x) \in [0,T] \times D$*

Definition 3.5.3. *The first order system* (3.82) *is called elliptic if the corresponding characteristic equation det $\| \, \sigma A_0(x,t) + \sum_{i=1}^{m} \xi A_i \, \| = 0$ has no real solution $(\sigma, \xi) \in \mathbb{R}^{m+1}$, $(\sigma,\xi) \neq 0$, $\forall \ (x,t) \in D \times [0,T]$.*

Example 3.5.1. *In the case of a second order PDE*

$$a(x,t)\partial_t^2 u + 2b(x,t)\partial_{(t,x)}^2 u + c(x,t)\partial_x^2 u + d(x,t)\partial_t u + e(x,t)\partial_x u = f(x,t,u), x \in \mathbb{R} \tag{3.83}$$

We get the following characteristic equation

$$a(x,t)\sigma^2 + 2b(t,x)\sigma\xi + c(t,x)\xi^2 = 0 \tag{3.84}$$

Hint Using a standard procedure we rewrite the scalar equation (3.83) as an evolution system for the unknown vector $u_1(t,x) = \partial_x u(t,x)$, $u_2(t,x) = \partial_t u(t,x), u_3(t,x) = u(t,x)$. We get

$$\begin{cases} \partial_t u_1 = \partial_x u_2 \\ a(x,t)\partial_t u_2 = -2b(x,t)\partial_x u_2 - c(x,t)\partial_x u_1 + f_1(x,t,u_1,u_2,u_3) \\ \partial_t u_3 = u_2\,,\ t \in \mathbb{R},\, x \in \mathbb{R} \end{cases} \tag{3.85}$$

where$f_1(x,t,u_1,u_2,u_3) = f(x,t,u_3) - e(x,t)u_1 - d(x,t)u_2$. The system (3.85) can be written as (3.82) using the following (3×3) matrices

$$A_0(x,t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a(x,t) & 0 \\ 0 & 0 & 1 \end{pmatrix},\ A_0(x,t) = \begin{pmatrix} 0 & -1 & 0 \\ c(x,t) & 2b(x,t) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Compute

$$\begin{aligned} det \parallel \sigma A_0(x,t) + \xi A_1(x,t) \parallel &= det \begin{pmatrix} \sigma & -\xi & 0 \\ \xi.c & \sigma a + 2b\xi & 0 \\ 0 & 0 & \sigma \end{pmatrix} \\ &= \sigma[a(x,t)\sigma^2 + 2b(x,t)\sigma\xi + c(x,t)\xi^2] \end{aligned}$$

and the characteristic equation (3.84) is obtained.

3.6 Adjoint Second Order Differential Operator; Riemann and Green Formulas

We consider the following linear second order differential operator

$$L\,\nu = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}\partial_{ij}^2\nu + \sum_{i=1}^{n} B_i\partial_i\nu + C\nu \tag{3.86}$$

where the coefficients $A_{ij},\, B_i,\, and\, C$ are second order continuously differentiable scalar functions of the variable $x = (x_1, ..., x_n) \in \mathbb{R}^n$ and $\partial_i\nu = \frac{d\nu}{dx_i},\, \partial_{ij}^2\nu = \frac{\partial^2\nu}{\partial x_i \partial x_j}$. Without restricting generality we may assume that $A_{ij} = A_{ij}(see A_{ij} \to \frac{1}{2}(A_{ij} + A_{ij}))$ and define the adjoint operator associated with $L(L^\infty = M)$

$$M\nu \equiv \sum_{i=1}^{n}\sum_{j=1}^{n} \partial_{ij}^2(A_{ij}\nu) - \sum_{i=1} \partial_i(B_i\nu) + C\nu \tag{3.87}$$

By a direct computation we convince ourselves that the following formula is valid

$$\nu L_u - uM\nu = \sum_{i=1}^{n} \partial_i P_i,\ u, \nu \in \mathcal{C}^2(\mathbb{R}^n) \tag{3.88}$$

where

$$P_i = \sum_{j=i}^{n} [\nu A_{ij}\partial_j u - u\partial_j(A_{ij}\nu)] + B_i u\nu$$

In this respect notice that

$$\begin{aligned}\sum_{i=1}^{n}\partial_i P_i &= \Big[\sum_{i=1}^{n}\sum_{j=1}^{n}\nu A_{ij}\partial_{ij}^2 u+\sum_{i=1}^{n}\nu B_i\partial_i u + Cu\nu\Big]\\ &- \Big[\sum_{i=1}^{n}\sum_{j=1}^{n}u\partial_{ij}^2(A_{ij}\nu)-\sum_{i=1}^{n}u\partial_i(B_i\nu)+Cu\nu\Big]\\ &+ \sum_{i=1}^{n}\sum_{j=1}^{n}[\partial_j u\partial_i(A_{ij}\nu)-\partial_i u\partial_j(A_{ij}\nu)]\end{aligned}$$

where the last term is vanishing.

3.6.1 Green Formula and Applications

Consider a bounded domain $\Omega \subseteq \mathbb{R}^n$and assume that its boundary $S = \partial\Omega$ can be expressed using piecewise first order continuously differentiable functions. Then using Gauss-Ostrogradsky formula associated with (3.88) we get

$$\int_{\Omega}(\nu Lu - uM\nu)dx_1, ...dx_n = +\int_{S}(\sum_{i=1}^{n}P_i cosnx_i)dS \tag{3.89}$$

(Green formula) where $cos\, n\, x_i$, $i \in \{1, ..., n\}$, are the coordinates of the unit orthogonal vector $\overrightarrow{n}$ at S oriented outside of S.

Example 3.6.1. *Assume $Lu = \Delta u = \sum_{i=1}^{3}\partial_i^2 u$(laplacian) associate the adjoint $M\nu = \Delta\nu$ (laplacian) and $P_1 = \nu\partial_i u - u\partial_1\nu, P_2 = \nu\partial_2 u - u\partial_2\nu, P_3 = \nu\partial_3 u - u\partial_3 u$ satisfying* (3.88). *Applying* (3.89) *we get the Green formula for Laplace operator.*

$$\begin{aligned}\iiint_{\Omega}(\nu\delta u - u\delta\nu)dx_1dx_2dx_3 &= \iint_{S}[P_1(cosnx_1)+P_2(cosnx_2)+P(cosnx_3)]dS\\ &= \iint_{S}[\nu\frac{\partial u}{\partial n}-u\frac{\partial u}{\partial n}]dS\end{aligned} \tag{3.90}$$

Where $\frac{\partial u}{\partial n}(\frac{\partial\nu}{\partial n})$ stands for the derivative in direction of the vector

$$\overrightarrow{n}\frac{\partial u}{\partial n} = \sum_{i=1}^{3}(\partial_i u)cosnx_i =< \partial_x u, \overrightarrow{n} > \overline{n} = (cosnx_1, (cosnx_2, (cosnx_3))$$

Example 3.6.2. *Consider*

$$L \equiv \partial_{xy}^2 u + a(x,y)\partial_x u + b(x,y)\partial_y u + c(x,y)u$$

and define

$$M\nu \equiv \partial_{xy}^2 \nu - \partial_x(a\nu) - \partial_y(b\nu) + c\nu, P_1 = \frac{1}{2}(\nu\partial_y u - u\partial_y \nu) + a\,u\,\nu$$

and

$$P_1 = \frac{1}{2}(\nu\partial_x u - u\partial_x \nu) + b\,u\,\nu$$

The corresponding Green formula lead us to

$$\begin{aligned} \iint\limits_{\Omega} [\nu\, L\, u - u\, M, \nu] dx\, dy &= \int_S \{[\frac{1}{2}(\nu\partial_y u - u\partial_y \nu) \\ &= a\,u\,\nu]cos(n,x) + [\frac{1}{2}(\nu\partial_x u - u\partial_x \nu) + a\,u\,\nu]cos(n,y)\} ds \end{aligned} \tag{3.91}$$

3.6.2 Applications of Riemann Function

Using (3.91) we will find the Cauchy problem solution associated with the equation

$$L\, u = F \tag{3.92}$$

and the initial conditions

$$u|_{y=\mu(x)} = \varphi_0(x),\ \partial_y\, u|y = \mu(x) = \varphi_1(x),\ x \in [a,b] \tag{3.93}$$

Here the curve $\{y = y(x)\}$ is first order continuously differentiable satisfying $\frac{d\mu(x)}{dx} < 0\,, x \in [a,b]$. The algorithm belongs to Riemann and using (3.93) we find $\partial_x\, u|_{y=\mu(x)}$ as follows. By direct derivation of $u|_{y=\mu(x)} = \varphi_0(x)$, we get

$$\begin{cases} \frac{d\varphi_0}{dx}(x) = \partial_x\, u(x,\mu(x)) + \partial_y\, u(x,\mu(x))\frac{d\mu(x)}{dx} \\ \partial_x\, u(x,\mu(x)) = \phi_0'(x) - \varphi_1(x)\mu'(x) \end{cases} \tag{3.94}$$

Rewrite the Green formula (3.91) using a domain $\Omega \subseteq \mathbb{R}^2$ as follows. where A and B are located on the curve $y = \mu(x)$ intersected with the coordinates lines of the fixed point $P = (x_0, y_0)$. Applying the corresponding Green formula (3.91) on the domain

PAB (triangle) we get

$$\begin{aligned}\iint\limits_{\Omega}(\nu Lu - uM\nu)dxdy &= \int_A^B [\frac{1}{2}(\nu\partial_y u - u\partial_y\nu) + a\,u\,\nu]dy \\ &- \int_A^B [\frac{1}{2}(\nu\partial_x u - u\partial_x\nu) + b\,u\,\nu]dx \\ &+ \int_B^P [\frac{1}{2}(\nu\partial_y u - u\partial_y\nu) + a\,u\,\nu]dy \\ &+ \int_A^P [\frac{1}{2}(\nu\partial_x u - u\partial_x\nu) + a\,u\,\nu]dx \qquad (3.95)\end{aligned}$$

Where $dy = cos(n,x)dS$ and $dx = -cos(n,y)dS$ and considering that dS is a positive measure. Rewrite the last two integrals from (3.95)

$$\begin{aligned}\int_B^P [\frac{1}{2}(\nu\partial_y u - u\partial_y\nu) + a\,u\,\nu]dy &= \frac{1}{2}u\nu|_B^P + \int_B^P (-u\partial_y\nu) + a\,u\,\nu]dy \\ &= \frac{1}{2}u\nu|_B^P + \int_B^P u(-\partial_y\nu) + a\,\nu]dy \qquad (3.96)\end{aligned}$$

and

$$\int_A^P [\frac{1}{2}(\nu\partial_x u - u\partial_x\nu) + a\,u\,\nu]dx = \frac{1}{2}u\nu|_A^P + \int_A^P u(-\partial_x\nu) + b\,\nu]dx \qquad (3.97)$$

these formulas lead us to the solution provided the following Riemann function $\nu(x, y; x_0, y_0)$ is used

$$M\nu = 0, \nu|_{x=x_0} = \exp\int_{y_0}^{y} a(x_0,\sigma)d\sigma,\ \nu|_{y=y_0} = \exp\int_{x_0}^{x} b(\sigma, y_0)d\sigma \qquad (3.98)$$

From (3.98) we see easily that

$$\begin{cases} \nu(x_0, y_0; x_0, y_0) = 1\, and\, \partial_y\nu|_{x=x_0} = a(x_0, y)\nu|_{x=x_0}, \\ \partial_x\nu|_{y=y_0} = b(x, y_0)\nu|_{y=y_0} \end{cases} \qquad (3.99)$$

Using (3.98) and (3.99) into (3.95) we get

$$\iint\limits_{\Omega} \nu(x, y; x_0, y_0)F(x, y)dxdy = (u\nu)(P) + \Gamma + I = u(x_0, y_0) + \Gamma + I \qquad (3.100)$$

where

$$\Gamma = +\int_A^B [\frac{1}{2}(\nu\partial_y u - u\partial_y\nu) + a\,u\,\nu]dy - \int_A^B [\frac{1}{2}(\nu\partial_x u - u\partial_x\nu) + b\,u\,\nu]dx \qquad (3.101)$$

and

$$I = -\frac{1}{2}[(u\nu)(B) + (u\nu)(A)] + \int_B^P u(-\partial_y \nu) + a\,\nu]dy + \int_A^P u(-\partial_x \nu) + b\,\nu]dx \quad (3.102)$$

Notice that the integral in (3.102) are vanishing provided that the initial conditions (3.99) are used for Riemann function ν (see(3.98) and (3.99)) and the equation (3.100) is written as

$$u(x_0, y_0) = \frac{1}{2}(u\nu)(B) + (u\nu)(A) - \Gamma + \iint_\Omega \nu\, F(x, y)dxdy \quad (3.103)$$

where Γ is defined in (3.101). The Cauchy problem solution defined in (3.92) and (3.93) has a more explicit expression if the Cauchy conditions are vanishing and it can be achieved by modifying the right hand side F as follows

$$\begin{aligned} F_1(x, y) &= F(x, y) - \varphi_1'(x) - a(x, y)\{\varphi_0'(x) + [y - \nu(x)]\varphi_1'(x) - \nu'(x)\varphi_1(x)\} \\ &= b(x, y)\varphi_1(x) - c(x, y)\{\varphi_0(x) + (y - \mu(x))\varphi_1(x)\} \end{aligned} \quad (3.104)$$

Theorem 3.6.1. *The Cauchy problem solution defined in* (3.92) *and* (3.93) *is given by*

$$u(x_0, y_0) = \iint_\Omega \nu(x, y; x_0, y_0)F_1(x, y)dxdy$$

where ν is the corresponding Riemann function defined in (3.98) *and* (3.99) *and F_1 is* (3.104).

Remark 3.6.2. *In the case we consider a rectangle $PASB = \Omega$ and look for a Cauchy problem solution*

$$L\,u = F \text{ (where } L \text{ and } F \text{ are defined in (3.92))} \quad (3.105)$$

$$u|_{x=x_1} = \phi_1(y), u|_{y=y_1} = \phi_2(x), S = (x_1, y_1) \quad (3.106)$$

then Green formula lead as to the following solution

$$u(x_0, y_0) = u|_p = u\nu|_S + \int_B^B \nu(\varphi_2' + b\varphi_2)dx + \int_S^A \nu(\varphi_1' + a\varphi_1)dy + \iint_\Omega \nu\, F(x, y)dxdy \quad (3.107)$$

where $\nu(x, y; x_0, y_0)$ is associated Riemann function satisfying (3.98) *and* (3.99). *Let $u(x, y; x_1, y_1)$ be the Riemann function for the adjoint equation $M\nu = 0$ satisfying*

$$Lu = 0, u|_{y=y_1} = exp \int_{x_1}^{x} b(\sigma_1, y_1)d\sigma, u|x = x_1 = exp \int_{y_1}^{y} a(x_1, \sigma)d\sigma \quad (3.108)$$

In this particular case , the corresponding solution u verify (3.105) *and* (3.106) *with*

$F = 0, \varphi_2' + by\varphi_2 = \varphi_1' a\varphi_1 = 0$, *and the representation formula* (3.107) *becomes*

$$u(x_0, y_0, x_1, y_1) = \nu(x_1, y_1, x_0, y_0)(\text{see } u(S) = u(x_1, y_1; x_1, y_1) = 1) \tag{3.109}$$

The equation (3.109) *tell us that the Riemann function is symmetric: if* $u(x_0, y_0, y_1; x_1, y_1)$ *is the Riemann function for*$(x_0, y_0) \in \mathbb{R}^2$ *satisfying* $(L\,u)(x_0, y_0) = 0$ *and initial condition given at* $x = x_1, y = y_1$ *then* $u(x_0, y_0; x_1, y_1)$ *as a function of* $((x_1, y_1)$ *verifies the adjoint equation* $Mv = 0$.

Remark 3.6.3. *The equation* (3.104) *reflects the continuous dependence of a solution for the Cauchy problem defined in* (3.92) *and* (3.93) *with respect to the right hand side* F *and initial conditions* φ_0, φ_1 *restricted to the triangle* PAB. *A solution is determined inside of this triangle using the fixed* F, φ_0, φ_1 *and the curve* $y = \mu(x)$ *and outside of this triangle. We may get different solutions according to the modified initial conditions prescribed outside of this triangle.*

Remark 3.6.4. *A unique solution (Riemann function) verifying the Cauchy problem defined in* (3.108) *can be obtained directly using the standard method of approximation for a system of integral equations associated with*(3.98). *In the rectangle* $x_0 \leqslant x \leqslant a\,, y_0 \leqslant y \leqslant b$. *we are given* $\nu|_{x=x_0} = \varphi_1(y), \nu|_{y=y_0} = \varphi_2(x)$ *with* $\varphi_1(y_0) = \varphi_2(x_0)$ *and denote* $\partial_x \nu = \lambda, \partial_y \nu = w$. *Rewrite* (3.98) *as follows*

$$\frac{\partial \lambda}{\partial y} = \frac{\partial W}{\partial x} = a(x,y)\lambda + b(x,y)W + \widetilde{C}(x,y)\nu$$

where $\widetilde{C}(x,y) = -c(x,y) + \partial_x a(x,y) + \partial_y b(x,y)$ *and we get a system of integral equations*

$$\begin{cases} \lambda(x,y) = \lambda(x,y_0) + \int\limits_{y_0}^{y} [a(x,\sigma)\lambda(x,\sigma) + b(x,\sigma)\omega(x,\sigma) + \widetilde{C}(x,\sigma)v(x,\sigma)]d\sigma \\ w(x,y) = w(x_0,y) + \int\limits_{x_0}^{x} [a(x,\sigma)\lambda(x,\sigma) + b(x,\sigma)\omega(x,\sigma) + \widetilde{C}(x,\sigma)v(x,\sigma)]d\sigma \\ \nu(x,y) = \varphi_2(x) + \int\limits_{y_0}^{y} \omega(x, y\sigma)d\sigma \end{cases}$$

where $\lambda(x, y_0) = \varphi_2'(x)\,, w(x_0, y) = \varphi_1'(y)$.

Example 3.6.3. *(Green functions for Lapace operator)*
Let $\Omega \in \mathbb{R}^3$ *be a bounded domain whose boundary* $S = \partial\Omega$ *is represented by smooth functions.We are looking for a smooth function* $u : \Omega \to \mathbb{R}$ *satisfying Poisson equation*

$$\Delta\, u = f(P),\ P \in \Omega(\ \Delta\text{-Laplace operator}) \tag{3.110}$$

and one of the following boundary conditions

$$u|_S = F_0(S)\,(\text{Dirichlet problem}) \tag{3.111}$$

$$\frac{\partial u}{\partial n}|_S = F_1(S)\,(\text{Neumann problem}) \tag{3.112}$$

It is known that Laplace operator is self adjoint and using Green formula

$$\iiint\limits_{\Omega} (\nu \Delta u - u \Delta \nu) d\Omega = \iint\limits_{S} (u \frac{\partial \nu}{\partial n} - \nu \frac{\partial u}{\partial n}) dS \tag{3.113}$$

we see that a vanishing Dirichlet (Neumann) condition is selfadjoint as the Green formula (3.113) *shows.*

3.6.3 Green function for Dirichlet problem

Definition 1
Let $G(P, P_0),\ P, P_0 \in \Omega$, be a scalar function satisfying

1. $G(P, P_0)$ is a harmonic function $(\Delta\, G = 0)$ with respect to $P \in \Omega,\ P \neq P_0$
2. $G(P, P_0)|_{P \in S} = 0,\ S = \partial\Omega$
3. $G(P, P_0) = \frac{1}{4\pi r} + g(P, P_0)$, where $r = |P - P_0|$

and g is a second order continuously differentiable function of $P \in \Omega$ verifying $\Delta\, g = 0,\ \forall\, P \in \Omega$.
A function $G(P, P_0)$ fulfilling 1, 2 and 3 is called a Green function for Dirichlet problem (3.110), (3.111). The existence of a Green function satisfying 1 , 2 and 3 is analyzed in [2](see [2] in references)

Theorem 3.6.5. *Assume that the Green function $G(P, P_0)$ satisfying 1, 2 and 3 is found such that the normal derivative $\frac{\partial G}{\partial n}(s)$ of G at each $s \in S$ exists. Then the solution of Dirichlet problem* (3.110)+(3.111) *is represented by*

$$u(P_0) - \iint\limits_{S} F_0(s) \frac{\partial G}{\partial n}(s) ds - \iiint\limits_{\Omega} G(P, P_0) f(P) dx dy dz$$

Proof. Take $\delta > 0$ sufficiently small such that $B(P_0, \delta) \subseteq \Omega$ and denote $\sigma = \partial B(P_0, \delta)$. Apply Green formula in the domain $\Omega' = \Omega \setminus B(P_0, \delta)$. Let $u(P),\ P \in \Omega$, be the solution of the Dirichlet problem (3.110)+(3.111) and denote $\nu(P) = G(P, P_0),\ P \in \Omega'$. Both functions $u(P), \nu(P), P \in \Omega'$, are continuously differentiable and we get

$$\iiiint\limits_{\Omega'} G(P, P_0) \Delta u dP = \iint\limits_{S'} (u \frac{\partial G}{\partial n} - G \frac{\partial u}{\partial n})(s) ds \tag{3.114}$$

where $S' = S \cup \sigma = \partial\Omega'$
Denote n the orthogonal vector at the surface $\sigma = \partial B(P_0, \delta),\ |n| = 1$, oriented to the

center P_0 and n' is the orthogonal vector at σ oriented in the opposite direction of n.Using $G|_{P\in S} = 0$ we get

$$\iiint\limits_{\Omega'} G(P,P_0)f(P)dx\,dy\,dz = \iint\limits_{S} F_0(s)\frac{\partial G}{\partial n}(s)ds - \frac{1}{4\pi}\iint\limits_{\sigma}(u\frac{\partial}{\partial n}\frac{1}{r} - \frac{1}{r}\frac{\partial u}{\partial n})d\sigma - \iint\limits_{\sigma}(u\frac{\partial g}{\partial n} - g\frac{\partial u}{\partial n})d\sigma \tag{3.115}$$

Letting $\delta \to 0$ we get that the last integral in (3.115) is vanishing (see u, g are continuously differentiable with bounded erivatives). In addition, the second integral in the right hand side of (3.115) will become.

$$\lim_{\delta\to}\frac{1}{4\pi}\iint\limits_{\sigma}(u\frac{\partial}{\partial n}|\frac{1}{r} - \frac{1}{r}\frac{\partial u}{\partial n})d\sigma = \lim_{\delta\to}\frac{1}{4\pi\delta^2}\iint\limits_{\sigma} ud\sigma - \lim_{\delta\to}\frac{1}{4\pi^2}\iint\limits_{\sigma}\delta\frac{\partial u}{\partial n}d\sigma = u(p_0) \tag{3.116}$$

$$u(p_0) = \iint\limits_{S} F_0(s)\frac{\partial u}{\partial n}(s)ds - \iiint\limits_{\Omega} G(p,p_0)f(p)dxdydz \tag{3.117}$$

and the proof is complete. □

Theorem 3.6.6. *The function $u(p_0), p_0 \in \Omega$ given in Theorem 3.6.5 is the solution of the Dirichlet problem* (3.110)+(3.111)*.*

Proof. It is enough to prove the existence of the Dirichlet problem solution (3.110)+(3.111). Let φ be the newtonian potential with the density function$f(p)$ on the domain Ω,

$$\varphi(p_0) = -\frac{1}{4\pi}\iiint\limits_{\Omega}\frac{1}{r}f(p)dxdydz \tag{3.118}$$

It is known that φ satisfies the following Poisson equation $\delta\varphi = f$. Define $\nu = u - \varphi$ and it has to satisfy

$$\begin{cases} \nu|S = u|S - \varphi = F_0(S) = \nu(s) \\ \Delta\nu(P) = 0, P \in \Omega(\nu(P_0) = \iint\limits_{S} v_0(s)\frac{\partial G}{\partial n}ds) \end{cases} \tag{3.119}$$

The existence of the harmonic function ν verifying (3.119) determines the unknown u as a solution of the Dirichlet problem (3.110)+(3.111). □

3.7 Linear Parabolic Equations

The simplest linear parabolic equation is defined by the following heat equation

$$\partial_t u(t,x) = \partial_x^2 u(t,x),\ t > 0, x \in \mathbb{R} \tag{3.120}$$

$$\lim_{t\to 0} u(t,x) = \varphi(x), x \in \mathbb{R} \tag{3.121}$$

$\varphi \in C_b(R)$ is fixed. A solution is a continuous function $u(t,x); [0,\infty)\times\mathbb{R} \to \mathbb{R}$ which is second order continuously differentiable of $x \in \mathbb{R}$ for each $t > o$ and satisfying (3.120) + (3.121) for some fixed continuous and bounded function $\varphi \in \mathcal{C}_b(\mathbb{R})$.

3.7.1 The Unique Solution of the Cauchy Problem (3.120) and (3.121)

It is expressed as follows.

$$u(t,x) = \frac{1}{\sqrt{4\pi t}} \int\limits_{-\infty}^{\infty} \varphi(\xi)\exp - \frac{(x-\xi)^2}{4t} d\xi \tag{3.122}$$

(Poisson formula for heat equation) Denote

$$P(\sigma,x,y)dy = (\sqrt{4\pi\sigma})^{-1}\exp - \frac{(y-x^2)}{4\sigma}, for\, \sigma > 0,\ x,y \in \mathbb{R} \tag{3.123}$$

By a direct computation we get the following equations

$$\int\limits_R P(\sigma,x,y)dy = 1 and \partial_\sigma P(\sigma,x,y) - \partial_x^2 P(\sigma,x,y) = 0\, for\, any\, \sigma > 0, x,y \in \mathbb{R} \tag{3.124}$$

The first equation of (3.124) is obtained using a change of variable $\frac{y-x}{2\sqrt{\sigma}} = z$ and

$$\int\limits_R P(\sigma,x,y)dy = \sqrt{4\sigma}\int\limits_R P(\sigma,x,x+2\sqrt{\sigma}z)dz = \frac{1}{\sqrt{\pi}}\int\limits_R (\exp - |z|^2)dt = 1 \tag{3.125}$$

where $\int\limits_R (\exp - t^2)dt = \sqrt{\pi}$ is used. In addition, the function $P(\sigma,x,y)$ defined in (3.123) fulfils the following.

$$\lim_{\sigma\to 0}\int_R \varphi(y)P(\sigma,x,y)dy = \varphi(x),\ for\, each\, x \in \mathbb{R} \tag{3.126}$$

where $\varphi \in \mathcal{C}_b(R)$ is fixed. Using (3.124) we see easily that $\{u(t,x) : t > 0, x \in \mathbb{R}\}$ defined in (3.122) satisfies heat equation (3.120), provided we notice that

$$u(t,x) = \int_R \varphi(\xi)P(t,x,\xi)d\xi,\ t > 0,\ x \in \mathbb{R} \tag{3.127}$$

and

$$\begin{cases} \partial_t\, u(t,x) = \int_R \varphi(\xi)\partial_t P(t,x,\xi)d\xi, \\ \partial_x^2\, u(t,x) = \int_R \varphi(\xi)\partial_x^2 P(t,x,\xi)d\xi, \end{cases} \tag{3.128}$$

The property (3.126) used for (3.127) allows one to get (3.121).

Remark 3.7.1. *The unique solution of the heat equation* (3.120) + (3.121) *can be expressed by the formula* (3.122) *even if the continuous* $\varphi(x) : \mathbb{R} \to \mathbb{R}$ *satisfies a polynomial growth condition*

$$|\varphi(x)| \leqslant c(1 + |x|^N),\ \forall\, x \in \mathbb{R} \tag{3.129}$$

There is no change in proving that the unique Cauchy problem solution of the heat equation for $x \in \mathbb{R}$.

$$\partial_t u(t,x) = \Delta_x\, u(t,x),\ t > 0,\ x \in \mathbb{R} \tag{3.130}$$

$$\lim_{t\to 0} u(t,x) = \varphi(x), x \in \mathbb{R}^n,\ \Delta_x = \sum_{i=1}^{n} \partial_{x_i}^2 \tag{3.131}$$

is expressed by

$$u(t,x) = (4\pi\, t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \varphi(\xi)\, exp - \frac{|x-\xi|^2}{4t} d\xi \tag{3.132}$$

where $\varphi \in \mathcal{C}_b(\mathbb{R}^n)$.

3.7.2 Exercises

(a_1) Find a continuous and bounded function $u(t,x); [0,T] \times R \to R$ fulfilling the following

$$\begin{cases} (1)\ \partial_t u(t,u) = a^2 \partial_x^2 u(t,x), t \in (0,T],\ x \in \mathbb{R},\ a > 0 \\ (2)\ \lim\limits_{t\to 0} u(t,x) = cos x,\ x \in \mathbb{R}. \end{cases}$$

Hint. The equation (1) can be written with the constant $a^2 = 1$ provided we use a change of variable $\xi = \frac{x}{a}$ and denote

$$u(t,x) = \nu(t, \frac{x}{a}),\ t \in [0,T], x \in \mathbb{R}$$

Here $\nu(t,y) : [0,T] \times \mathbb{R} \to \mathbb{R}$ satisfies the equation

$$\partial_t \nu(t,y) = \partial_y^2 \nu(t,y),\ (t,y) \in (0,T] \times \mathbb{R}$$

and initial condition $\nu(0,y) = cos(ay)$ (a_2). Find a continuous and bounded function $u(t,x,y) : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfying the following linear parabolic equation

$$\begin{cases} \textbf{(1)}\, \partial_t u(t,x,y) = a^2 \partial_x^2 u(t,x,y) + b^2 \partial_y^2 u(t,x,y),\ t \in (0,T] \\ \textbf{(2)}\, \lim\limits_{t\to 0} u(t,x,y) = sinx + cosy,\ a > 0,\ b > 0,\ x,y \in \mathbb{R} \end{cases}$$

Hint. The equation (1) and (2) will be rewritten using the following changes $y_1 = \frac{x}{a}$, $y_2 = \frac{y}{b}$ and
$u(t,x,y) = \nu(t, \frac{x}{a}, \frac{y}{b})$ where $\nu(t,y_1,y_2) : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfy (see(3.130)$n = 2$)

$$\begin{cases} \partial_t \nu = \partial_{y_1}^2 \nu + \partial_{y_2}^2 \nu \\ \lim\limits_{t\to 0} \nu(t,y_1,y_2) = sin(ay_1) + cos(by_2) \end{cases}$$

3.7.3 Maximum Principle for Heat Equation

Any continuous solution $u(t,x)$, $(t,x) \in [0,T]\times[A,B]$, satisfying heat equation (3.120) for $0 < t \leqslant T$ and $x \in (A,B)$ will achieve its extreme values
$\max_{(t,x)\in[0,T]\times[A,B]} u(t,x)$ and $\min_{(t,x)\in[0,T]\times[A,B]} u(t,x)$ on the boundary $\widehat{\partial D}$ of the domain $D = \{(t,x) \in [o,T] \times [A,B]\}$, where

$$\widehat{\partial D} = (\{0\} \times [A,B]) \bigcup ([0,T] \times \{A\}) \bigcup ([0,T] \times \{B\})$$

Proof. Denote $M = \max_{(t,x)\in D} u(t,x)\,, m = \max_{(t,x)\in\widehat{\partial D}} u(t,x)$ and assume $M > m$. Let $(t_0,x_0) \in D$, $(t_0,x_0)\, not\, in\, \widehat{\partial D}$ be such that $u(t_0,x_0) = M$ and consider the following auxiliary function

$$\nu(t,x) = u(t,x) + \frac{M-m}{2(B-A)^2}(x-x_0)^2 \tag{3.133}$$

We see easily that $\nu(t,x)$ satisfies

$$\nu(t,x) \leqslant u(t,x) + \frac{M-m}{2(B-A)^2}(B-A)^2 \leqslant m + \frac{M-m}{2} < M \tag{3.134}$$

for any
$(t,x) \in \widehat{\partial D}$, and $\nu(t_0,x_0) = u(t_0,x_0) = M$, consider $\nu(t_1,x_1) = \max_{(t,x)\in D} \nu(t,x) \geqslant M$. As a consequence, (t_1,x_1) not in $\widehat{\partial D}$ and

$$\begin{cases} \partial_t \nu(t_1,x_1) = 0,\ \partial_x \nu(t_1,x_1) = 0,\ \partial_x^2 \nu(t_1,x_1) \leqslant 0,\ if\ (t_1,x_1) \in intD \\ \partial_t \nu(t_1,x_1) \geqslant 0,\ \partial_x \nu(t_1,x_1) = 0,\ \partial_x^2 \nu(t_1,x_1) \leqslant 0,\ if\ (t_1,x_1) \in \{T\} \times (A,B) \end{cases} \tag{3.135}$$

In both cases $(\partial_t \nu - \partial_x^2 \nu)(t_1,x_1) \geqslant 0$ and (t_1,x_1) not in $\widehat{\partial D}$. On the other hand,using

the heat equation (3.120) satisfied by $u(t,x)$ when (t,x) not in $\widehat{\partial D}$ we get (see(3.133))

$$(\partial_t \nu - \partial_x^2 \nu)(t_1, x_1) = -\frac{M-m}{(B-A)^2} < 0 \qquad (3.136)$$

contradicting the above given inequality. It proves that$M \leqslant m$. Replacing u with $\{-u\}$ and using

$$\max_{(t,x)\in D} \{-u(t,x)\} = \max_{(t,x)\in \widehat{\partial D}} \{-u(t,x)\}$$

we get the second conclusion

$$\min_{(t,x)\in D} \{-u(t,x)\} = \min_{(t,x)\in \widehat{\partial D}} \{-u(t,x)\}$$

$\square$

Remark 3.7.2. *In the above given proof we may assume that the heat equation* (3.120) $\partial_t u(t,x) = \partial_x^2 u(t,x)$ *is satisfied for any* $(t,x) \in (0,T) \times (A,B)$ *(omitting* $t = T$*) and the result is still valid noticing that* $u(t,x) \leqslant m$ *for* $0 \leqslant t \leqslant T - \varepsilon$ *will imply* $u(t,x) \leqslant m$ *for any* $0 \leqslant t \leqslant T$ *(see* u *is continuous). In addition, the conclusion of the maximum principle allows to extend it for* $|u(t,x)|$, *i.e* $|u(t,x)| \leqslant max\{|u(t,x)| : (t,x) \in \widehat{\partial D}\}$.

Remark 3.7.3. *The computation and arguments used for the scalar heat equation* (3.120) *can be extended to the case* $x \in \mathbb{R}^n$ *replacing the equation* (3.120) *by* (3.130) *and considering a ball* $B(P_*, \rho) \subseteq \mathbb{R}^n$ *instead of* $[A, B] \subseteq \mathbb{R}$. *If it is the case, the corresponding maximum principle associated with heat equation ((*3.130) *says*

$$\max_{(t,x)\in D} u(t,x) = \max_{(t,x)\in \widehat{\partial D}} u(t,x)$$

and

$$\min_{(t,x)\in D} u(t,x) = \min_{(t,x)\in \widehat{\partial D}} u(t,x)$$

where
$D = [0,T] \times B(P_*, \rho)$, $\widehat{\partial D} = (\{0\} \times B(P_*, \rho)) \bigcup ([0,T] \times \partial B)$, $P_* \in \mathbb{R}^n, (fixed), \partial B = boundary\, of\, B(P_*, \rho)$.
Here the heat equation(3.130) *is assumed on the domain* $(t,x) \in (0,T] \times int\, B(P_0, \rho)$ *and the corresponding auxiliary function is given by(see*(3.133)*)*

$$\nu(t,x) = u(t,x) + \frac{M-m}{8\rho^2}|x - x_0|^2,$$

where

$$u(t_0, x_0) = M = \max_{(t,x)\in D} u(t,x) = m = \max_{(t,x)\in \widehat{\partial D}} u(t,x)$$

and $M = u(t_0, x_0) > m$ *will lead us to a contradiction.*

Remark 3.7.4. *Using maximum principle for heat equation we get that the Cauchy problem solution for* (3.120)+(3.121)*(or* (3.130)+(3.131)*)) is unique provided the initial condition* φ *and the solution* $\{u(t,x) : t \geqslant 0, x \in \mathbb{R}^n\}$ *are restricted to the bounded*

continuous functions. In this respect,let $M > 0$ be such that

$$|u(t,x)| \leqslant M,\, |\varphi(x)| \leqslant M\, for\, any\, t \geqslant 0,\, x \in \mathbb{R}(x \in \mathbb{R}^n)$$

Consider the following particular solution of (3.120)

$$\nu(t,x) = \frac{2M}{L^2}(x^2 + 2t)\;\; satisfying\; \partial_t \nu = \partial_x^2 \nu\, and\, \nu(0,x) = \frac{2M}{L^2}x^2 \geqslant 0$$

$$\nu(\underset{-}{+}L, t) = \frac{2M}{L^2}(L^2 + 2t) \geqslant 2M$$

If (3.120)+(3.121) *has two bounded solutions then their difference $u(t,x) = u_1(t,x) - u_2(t,x)$ is a solution of heat equation* (3.120), *satisfying $|u(t,x)| \leqslant 2M$, $(t,x) \in [0,\infty) \times \mathbb{R}$,and $u(0,x) = 0$.On the other hand $h(t,x) = \nu(t,x) - u(t,x)$ satisfies* (3.120) *for any $(t,x) \in int\, D$, $D = [-L, L] \times [0,T]$ and*

$$h(t,x) = \nu(t,x) - u(t,x) \geqslant 0,\, \forall\, (t,x) \in \widehat{\partial D}$$

Using a maximum principle we get

$$u(t,x) \leqslant \frac{2M}{L^2}(x^2 + 2t)\, \forall\, (t,x) \in [0,T] \times [-L, L]$$

and similarly for $\overline{u}(t,x) = -u(t,x)$ we obtain

$$-u(t,x) \leqslant \frac{2M}{L^2}(x^2 + 2t)\, \forall\, (t,x) \in [0,T] \times [-L, L]$$

Combining the last two inequalities we get

$$|u(t,x)| \leqslant \frac{2M}{L^2}(x^2 + 2t)\, \forall\, (t,x) \in [0,T] \times [-L, L]$$

and for any arbitrary fixed $(t,x)(t > 0)$ letting $L \uparrow \infty$ we obtain $u(t,x) = 0$ which proves $u_1(t,x) = u_2(t,x)$ for each $t > 0$.

Exercise. Use the above given algorithm($n = 1$)for the multidimensional heat equation (11)($n \geqslant 1$) and get the conclusion:the Cauchy problem solution of (3.130) + (3.131) is unique provided initial condition φ and the solutions$\{u(t,x) : t \geqslant 0, x \in \mathbb{R}\}$ are restricted to the bounded continuous functions.
Hint. Let $M > 0$ be such that $|u(t,x)|, |\varphi(x)| \leqslant M$ for any $t \geqslant 0, x \in \mathbb{R}^n$ and consider $V(t, x = \frac{2M}{L^2})(|x|^2 + 2t)$ satisfying (3.130) and (3.131) with $V(0,x) = \frac{2M}{L^2}|x|^2 \geqslant 0(V(t,x) = \frac{2M}{L^2})(|x|^2 + 2t) \geqslant 2M\, if\, x \in \partial B(0, L)$. Proceed as in Remark (3.7.4).
Problem P_1(Maximum Principle for Linear Elliptic Equation)

Consider the following linear elliptic equation

$$\begin{aligned}0 &= \sum_{i,j=1}^{n} a_{ij}\partial^2_{x_i x_j} u(x)\\ &= \text{Trace}[A.\partial^2_x u(x)]\\ &= \text{Trace}[\partial^2_x u(x)A],\ x \in \Omega \text{ (bounded domain) } \subseteq \mathbb{R}^n \qquad (3.137)\end{aligned}$$

where the symmetric matrix $A = (a_{i,j\in\{1,\ldots,n\}})$ is strictly positive definite($< x, A\,x >\geqslant \delta \parallel x \parallel^2,\ \forall\, x \in \mathbb{R}^n,\ for\, some\, \delta > 0$). Under the above given conditions, using an adequate transformation of coordinates and function, we get a standard Laplace equation in $\mathbb{R}^n$ for which the maximum principle is valid.
(R) Show that for a continuous and bounded function $u(x) : \overline{\Omega} \to \mathbb{R}$ satisfying (3.137) for any $x \in \Omega$ we get the following maximum principle:

$$\max_{x\in\overline{\Omega}} u(x) = \max_{x\in\Gamma} u(x)\,(\min_{x\in\overline{\Omega}} u(x) = \min_{x\in\Gamma} u(x)),\ where\,\overline{\Omega} = \Omega \sqcup \Gamma,\ \Gamma = \partial\Omega$$

Hint. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be and orthogonal matrix $(T^* = T^{-1})$ such that

$$A = T\,D\,T^{-1},\ where\,D = \text{diag}(d_1, \ldots, d_n),\ d_i > 0 \qquad (3.138)$$

Define $A^{1\backslash 2}$(square root of the matrix A)$= T\,D^{\frac{1}{2}}T^{-1}$ and make the following transformations

$$x = A^{1\backslash 2}y,\ \nu(y) = u(A^{1\backslash 2}y) \qquad (3.139)$$

Then notice that$\{\nu(y) : y \in \overline{\Omega_1}\}$ is a harmonic function satisfying

$$0 = \Delta\nu(y) = Trace[\partial^2_y \nu(y)],\ \forall\, y \in \Omega_1 \qquad (3.140)$$

where

$$\Omega_1 = [A^{1\backslash 2}]^{-1}\Omega \text{and} \Gamma_1 = \partial\Omega_1 = [A^{1\backslash 2}]^{-1}\Gamma$$

As far as the maximum principle is valid for $\nu(y) : \overline{\Omega_1} \to \mathbb{R}$ satisfying (3.140)we get that

$$u(x) = \nu([A^{1\backslash 2}]^{-1}x),\ x \in \overline{\Omega}$$

satisfies the maximum principle too.

Problen P_2(Maximum Princie for Linear Parabolic Equations)

Consider the following linear parabolic equation

$$\partial_t u(t,x) = \sum_{i,j=1}^{n} a_{ij}\partial^2_{x_i x_j} u(t,x) = Trace[A.\partial^2_x u(t,x)] = Trace[\partial^2_x u(t,x)A] \qquad (3.141)$$

$t \in (0,T]\,, x \in \Omega(bounded\,domain) \subseteq \mathbb{R}^n$ and let $u(t,x) : [0,T] \times \overline{\Omega} \to \mathbb{R}$ be a continuous and bounded function satisfying the parabolic equation(3.141). If the matrix $A = (a_{ij})_{i,j\in\{1,\ldots,n\}}$ is symmetric and strictly positive($< x, A\,x >\geqslant \delta \parallel x \parallel^2$ $,\ \forall\, x \in \mathbb{R}^n,\ for\, some\, \delta > 0$)then $\{u(x) : x \in \overline{\Omega}\}$ satisfies the following maximum

principle:

$$\max_{(t,x)\in D} u(t,x) = \max_{(t,x)\in\widehat{\partial D}} u(t,x)\,(\min_{(t,x)\in D} u(t,x) = \min_{(t,x)\in\widehat{\partial D}} u(t,x)) \tag{3.142}$$

where $D = [0,T]\times\overline{\Omega} \subseteq \mathbb{R}^{n+1}$ *and* $\widehat{\partial D} = (\{0\}\times\overline{\Omega}) \sqcup ([0,T]\times\partial\Omega)$.
Hint. The verification is based on the canonical form we may obtain in the right hand side of (3.141) provided the following transformations are performed

$$x = A^{1\backslash 2}y,\ \nu(t,y) = u(t, A^{1\backslash 2}y),\ y\in\Omega_1 = [A^{1\backslash 2}]^{-1}\Omega \tag{3.143}$$

where the square root of a symmetric and positive matrix $A^{1\backslash 2} = T(\Gamma)^{\frac{1}{2}}T^{-1}$ is used. Here $\Gamma = \mathrm{diag}(\gamma_1,...,\gamma_n)$, $\gamma_i > 0$ and $T:\mathbb{R}^n\to\mathbb{R}^n$ is an orthogonal matrix ($T^* = T^{-1}$) such that $A = T\,\Gamma\,T^{-1}$. Notice that using (3.141) we get that $\{\nu(t,x) : t\in[0,T], y\in\Omega_1\}$ satisfies the standard heat equation

$$\partial_t\nu(t,y) = \Delta_y\nu(t,y) = Trace[\partial_y^2\nu(t,y)],\ t\in[0,t],\ y\in\Omega_1 \tag{3.144}$$

and the corresponding maximum principle

$$\max_{(t,y)\in D_1} \nu(t,x) = \max_{(t,y)\in\widehat{\partial D_1}} \nu(t,x)\,(\min_{(t,y)\in D_1} \nu(t,x) = \min_{(t,y)\in\widehat{\partial D_1}} \nu(t,x)) \tag{3.145}$$

is valid,where $D_1 = [0,T]\times\overline{D_1}$ *and* $\widehat{\partial D_1} = (\{0\}\times\overline{\Omega_1})\sqcup([0,T]\times\partial\Omega_1)$. Using $u(t,x) = \nu(t,[A^{1\backslash 2}]^{-1}x)$, $x\in\Omega$, and (3.145) we obtain the conclusion (3.145).

3.8 Weak Solutions(Generalized Solutions)

Separation of Variables(Fourier Method)
Boundary problems for parabolic and hyperbolic PDE can be solved using Fourier method.We shall confine ourselves to consider the following two types of PDE

$$(I)\quad \frac{1}{a^2}\partial_t u(t,x,z) = \Delta u(t,x,y,z),\ t\in[0,T],\ (x,y,z)\in D\subseteq\mathbb{R}^3$$

$$(II)\quad \partial_t^2 u(t,x,z) = \Delta u(t,x,y,z),\ t\in[0,T],\ (x,y,z)\in D\subseteq\mathbb{R}^3$$

where the bounded domain D has the boundary $S = \partial D$. The parabolic equation (I) is augmented with initial conditions(Cauchy conditions)

$$(I_a)\qquad u(0,x,y,z) = \varphi(x,y,z),\ (x,y,z)\in D$$

and the boundary conditions

$$(I_b) \qquad u(t,x,y,z)|_{(x,y,z)\in S} = 0 \ \ t \in [0,T]$$

The hyperbolic equation (II) is augmented with initial conditions

$$(II_a) \qquad u(0,x,y,z) = \varphi_0(x,y,z),\ \partial_t u(0,x,y,z) = \varphi_1(x,y,z),\ (x,y,z) \in D$$

and the boundary conditions

$$(II_b) \qquad \frac{\partial u}{\partial n}(t,x,y,z) = (\omega_1 \partial_x u + \omega_2 \partial_y u + \omega_3 \partial_z u)|(x,y,z) \in S = 0$$

where $n = (\omega_1, \omega_2, \omega_3)$ is the unit orthogonal vector at S oriented outside of D.

3.8.1 Boundary Parabolic Problem

To solve the mixed problem (I, I_a, I_b) we shall consider the particular solutions satisfying (I, I_a, I_b) in the form

$$u(t,x,y,z) = T(t)U(x,y,z) \tag{3.146}$$

It lead us directly to the following equations

$$\frac{\Delta U(x,y,z)}{U(x,y,z)} = \frac{1}{a^2}\frac{T'(t)}{T(t)},\ t \in [0,T],\ (x,y,z) \in D \tag{3.147}$$

and it implies that each term in (3.147) equals a constant $-\lambda$ and we obtain the following equations

$$\Delta U + \lambda U = 0; T'(t) + a^2 \lambda T(t) = 0 \tag{3.148}$$

(see $T(t) = C\exp - \lambda a^2 t, t \in [0,T]$). Using (3.148) and the boundary conditions (I_b) we get

$$U(x,y,z)|_{(x,y,z)\in S} = 0 \tag{3.149}$$

The values of the parameter λ for which (3.148)+(3.149) has a solution are called eigenvalues associated with linear elliptic equation $\Delta U + \lambda U = 0$ and boundary condition (3.149). The corresponding eigenvalues are found provided a Green function is used which allow us to rewrite the elliptic equation as a Fredholm integral equation. Recall the definition of a Green function.

Definition 3.8.1. *Let $S = \partial D$ be defined by second order continuously differentiable function.A Green function for the Dirichlet problem $\Delta U = f(P)$, $U|_S = F_0(S)$ is a symmetric function $G(P,P_0)$ satisfying the following conditions with respect to $P \in D$*
$(\alpha)\, \Delta G(P,P_0) = 0,\ \forall\, P \in D, P \neq P_0$,where $P_0 \in D$ is fixed,
$(\beta)\, G(P,P_0)|_S = 0,\ G(P,P_0) = G(P_0,P)$, and $G(P,P_0) = \frac{1}{4\pi r} + g(P,P_0)$, $r = |P - P_0|$
$(\gamma)\, g(P,P_0)$ is second order continuously differentiable and $\Delta g(P,P_0) = 0,\ \forall\, P \in D$.
$(\delta)\, U(P_0) = \iint\limits_S F_0(s)\partial_n G(s) ds - \iiint\limits_D G(P,P_0) f(P) dxdydz$ where $f(P) = -\lambda U(P)$

and $F_0(s) = 0, s \in S$

The solution of the Dirichlet problem (3.148)+(3.149) can be expressed as in Theorem 3.6.5. Under these conditions,the integral representation formula (δ) lead us to the following Fredholm integral equation

$$U(P_0) = \lambda \iiint\limits_D G(P, P_0)U(P)dxdydz \tag{3.150}$$

where $G(P, P_0) = \frac{1}{4\pi r} + g(P, P_0)$ is an unbounded function (see $\frac{1}{r}$) verifying

$$G(P, P_0) \leqslant \frac{1}{r^\alpha},\, 0 < \alpha < 3,\, for\, some\, constant\ A > 0 \tag{3.151}$$

Define

$$G^*(P, P_0) = \min(G(P, P_0), \frac{A}{\delta^\alpha}),\, where\, \delta > 0\ is\ fixed$$

We get $G(P, P_0) - G^*(P, P_0) \geqslant 0\ \forall\, P \in D$ and

$$0 \geqslant G(P, P_0) - G^*(P, P_0) \leqslant A(\frac{1}{r^\alpha} - \frac{1}{\delta^\alpha})\ if\ r \leqslant \delta \tag{3.152}$$

Using (3.152) and δ sufficiently small we obtain

$$\begin{aligned} \iiint\limits_D |G(P, P_0) - G^*(P, P_0)|dP &= \iiint\limits_D [G(P, P_0) - G^*(P, P_0)]dP \\ &\leqslant A \iiint\limits_{\{r \leqslant \delta\}} \frac{1}{r^\alpha} dP \leqslant \frac{\varepsilon}{2} \text{ where } \varepsilon > 0 \end{aligned} \tag{3.153}$$

is arbitrarily fixed.On the other hand, $G^*(P, P_0)$ is a continuous and bounded function for $P \in D$ and approximate it by a degenerate kernel

$$G^*(P, P_0) = \sum_{i=1}^{N} \varphi_i(P)\psi_i(P_0) + G_2(P, P_0) \tag{3.154}$$

where

$$\iiint\limits_D |G_2(P, P_0)|dP \leqslant \frac{\varepsilon}{2}$$

From (3.153) and (3.154) we get that $G(P, P_0)$ in (3.150) can be rewritten as

$$G(P, P_0) = \sum_{i=1}^{N} \varphi_i(P)\psi_i(P_0) + G_1(P, P_0) \tag{3.155}$$

where $G_1(P,P_0) = G_2(P,P_0) + [G(P,P_0) - G^*(P,P_0)]$ satisfies

$$\iiint_D |G_1(P,P_0)|dP \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \tag{3.156}$$

Using these remarks we replace the equation (3.150) by the following one

$$\begin{aligned} U(P_0) - \lambda \iiint_D G_1(P,P_0)U(P)dP &= [(E - \lambda A_1)U](P_0) \\ &= \lambda \sum_{i=1}^{N} \psi_i(P_0) \iiint_D \varphi_i(P)U(P)dP \end{aligned} \tag{3.157}$$

where the operator $B_1\varphi = [E - \lambda A_1](\varphi)$ has an inverse

$$B_1^{-1} = [E + \lambda A_1 + \lambda^2 A_1^2 + ... + \lambda^k A_1^k + ...], \text{ for any } |\lambda| < \frac{1}{\| A_1 \|}, \text{ where } \| A_1 \| \leqslant \frac{\varepsilon}{2} \tag{3.158}$$

is acting from $\mathcal{C}(D)$ to $\mathcal{C}(D)$.Denote $\xi_i(P_0) = (B_1^{-1}\psi_i)(P_0)$ and rewrite (3.157) as the following equation

$$U(P_0) = \lambda \sum_{i=1}^{N} \xi_i(P_0) \iiint_D \varphi_i(P)U(P)dP \tag{3.159}$$

which has a nontrivial solution for any $\lambda \in \{\lambda_1, \lambda_2, ...\}$ where the sequence $\{\lambda_j\}_{j\geqslant 1}$ of real numbers satisfies $|\lambda| \leqslant C$ only for a finite terms,for each constant $C > 0$ arbitrarily fixed. Let$\{U_j\}_{j\geqslant 1}$ be a sequence of solutions associated with equation (3.159) and eigenvalues $\{\lambda_j\}_{j\geqslant 1}$, they are called eigen functions. Notice that (see (β)) $G(P,P_0)$ is a symmetric function which allows one to see that the eigenvalues are positive numbers, $\lambda_j > 0$, and the corresponding eigenfunctions $\{U_j\}_{j\geqslant 1}$ can be taken such that

$$\iiint_D U_i(P)U_j(P)dP = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \tag{3.160}$$

$\{U_j\}_{j\geqslant 1}$ is a complete system in $\widehat{C}(D) \subseteq C(D)$ i.e any $\varphi \in \widehat{C}(D)$ can be represented

$$\varphi(P) = \sum_{j=1}^{\infty} a_j U_j(P) \tag{3.161}$$

where the series is convergent in $L_2(D)$
$(see \sum_{j=1}^{\infty} a_j^2 < \infty)$ and

$$\lim_{N\to\infty} (\iiint_D |\sum_{j=1}^{N} a_j U_j(P) - \varphi(P)|^2 dP)^{1\backslash 2} = 0$$

The coefficients $\{a_j\}_{j\geqslant 1}$ describing the continuous function $\varphi \in \widehat{C}(D)$ are called Fourier coefficients and satisfy

$$\iiint_D \varphi(P)U_j(P)dP = a_j,\ j \geqslant 1,\ \varphi \in C(D) \tag{3.162}$$

is fixed in I_a. Now we are in position to define a weak solution of mixed problem (I, I_a, I_b) and it will be given as the following series

$$u(t,x,y,z) = \sum_{i=1}^{\infty}(\exp - \lambda_i a^2 t)a_i U_i(x,y,z) \tag{3.163}$$

which is convergent in $L_2(D)$, uniformly with respect to $t \in [0,T]$. Using (3.162) and (3.161) the initial condition in (I_a) is satisfied in a weak sense, i.e

$$\lim_{N\to\infty} \parallel u_N(0,P) - \varphi(.) \parallel_2 = 0, u_N(t,x,y,z) = \sum_{i=1}^{\infty}(\exp - \lambda_i a^2 t)a_i U_i(x,y,z)$$

Here $\{u_N(t,x,y,z) : t \in [0,T],\ (x,y,z) \in D\}_{N\geqslant 1}$ is defined as a sequence of solutions

$$u_N(t,x,y,z) = \sum_{i=1}^{N}(\exp - \lambda_i a^2 t)a_i U_i(x,y,z) \tag{3.164}$$

satisfying the parabolic equation (3.146), the boundary condition (I_b) and

$$u_N(0,x,y,z) = \varphi_N(x,y,z),\ (x,y,z) \in D \tag{3.165}$$

such that, $\lim_{N\to\infty} \parallel \varphi - \varphi_N \parallel_2 = 0$ ((I_a) is weakly satisfied).

3.8.2 Boundary Hyperbolic Problem

To solve the mixed problem (II, II_a, II_b) we shall proceed as in the parabolic case and look for a particular solution

$$u(t,P) = T(t)U(P),\ u \in \mathcal{C}^2(D),\ T \in \mathcal{C}^2([0,T]) \tag{3.166}$$

satisfying (II) and the boundary condition(II_b). The function (3.166) satisfies (II) if

$$T(t)\,\Delta U(P) = U(P)T''(t)\ or\ \frac{T''(t)}{T(t)} = \frac{\Delta\, U(P)}{U(P)} = -\lambda^2(const) \tag{3.167}$$

which imply the equations

$$T''(t) + \lambda^2\, T(t) = 0 \tag{3.168}$$

$$\Delta U(P) + \lambda^2 U(P) = 0,\ \frac{\partial U}{\partial n}|_S = 0 \tag{3.169}$$

For the Neúmann problem solution in (3.169), we use a Green function $G_1(P, P_0)$ satisfying

$$\begin{cases} \Delta\, G_1 = \frac{1}{C}(where\ C = vol\, D)\, P \neq P_0 \\ \partial_n G_1|_S = 0 \end{cases} \tag{3.170}$$

In this case $\frac{1}{C}$ stands for the solution of the adjoint equation

$$\Delta\, \psi = 0,\ \partial_n \psi|_S = 0 \tag{3.171}$$

The Green function G_1 has the structure

$$G_1(P, P_0) = \frac{1}{4\pi r} + g(P),\ r = |P, P_0|,\ g(P) = \alpha |P|^2 + g_1(P) \tag{3.172}$$

where g_1 verifies

$$\begin{cases} \Delta\, g_1(P) = 0,\ \forall\, P \in D, and \\ \partial_n\, g_1|_S = -[\alpha \partial_n |P|^2 + \frac{1}{4\pi}\, \partial_n(\frac{1}{r})]|_S \end{cases} \tag{3.173}$$

The constant α is found such that

$$\begin{aligned} \iiint\limits_D (\Delta\, G_1) dxdydz &= \frac{1}{4\pi} \iiint\limits_D \Delta(\frac{1}{r}) dxdydz + 6\alpha\, vol(D) \\ &= -1 + 6\alpha\, vol(D) \Rightarrow \alpha = 1 \setminus 3val(D) \end{aligned} \tag{3.174}$$

In addition, using the Green function G_1 we construct a weak solution for the Neúmann problem

$$\Delta U(P) = f(P),\ \ (\partial_n U)|S = 0 \tag{3.175}$$

assuming that

$$\iiint\limits_D f(P) dxdydz = 0 \tag{3.176}$$

The Green formula (3.149) used for $\nu = G_1$ and $\{U(P : P \in D)\}$ satisfying (3.175) and (31) will get the form

$$\iiint\limits_D u(P) \Delta\, G_1(P, P_0)) dxdydz = \iiint\limits_D G_1(P, P_0) f(P) dxdydz \tag{3.177}$$

Looking for solution of (3.169) which verify

$$\iiint\limits_D U(P) dxdydz = 0 (see\ f(P) = -\lambda^2\, U(P)\, in\ (30)) \tag{3.178}$$

from (3.170) and (3.177) we obtain an integral equation

$$U(P_0) = \lambda^2 \iiint\limits_D G_1(P, P_0)U(P)dxdydz,\ P = (x, y, z) \tag{3.179}$$

which has a symmetric kernel $G_1(P, P_0)$. As in the case of the parabolic mixed problem we get a sequence of eigenvalues and the corresponding eigenfunctions $\{U_j\}_{j\geqslant 1}$. In this case they are satisfying

$$U_j \in \mathcal{C}^2(D),\ \partial_n U_j|_S = 0\, and\ \iiint\limits_D U_j(P)dxdydz = 0,\ j \geqslant 1 \tag{3.180}$$

Define the space $\widehat{C_0}(D) \subseteq \widehat{C}^1(D)$ consisting from all continuously differentiable functions $\varphi \in \mathcal{C}^1(D)$ verifying

$$(\partial_n\varphi)(P \in S) = 0, \iiint\limits_D \varphi(P)dxdydz = 0\, and\, \varphi(P) = \sum_{j=1}^{\infty} a_j U_j(P),\ \sum_{j=1}^{\infty} |a_j|^2 < \infty \tag{3.181}$$

The boundary problem $((II),(II_b))$ has two independent solutions (see $T_1(t) = cos\lambda_j t, T_2(t) = sin\lambda_j t$)

$$U_j(P)cos\lambda_j t\, and\, U_j(P)sin\lambda_j t,\ foreach\, j \geqslant 1 \tag{3.182}$$

and we are looking for a solution of the mixed problem $((II),(II_a),(II_b))$ as a convergent series

$$U(t, P) = \sum_{j=1}^{\infty}[a_j U_j(P)cos\lambda_j t + b_j U_j(P)sin\lambda_j t] + b_0 t \tag{3.183}$$

in $L_2(D)$ with respect to $P = (x, y, z)$ and uniformly with respect to $t \in [0, T]$. Here $\{a_j\}_{j\geqslant 1}$ must be determined as the Fourier coefficients associated with initial condition $\varphi_0 \in \widehat{C}_0(D)$

$$\varphi_0(P) = u(0, P) = \sum_{j=1}^{\infty} \alpha_j u_j(P) \tag{3.184}$$

and $\{b_j\}_{j\geqslant 0}$ are found such that the second initial conditions $\partial_t u(0, P) = \varphi_1(P)$ (see$(II_a) and \varphi_1 \in \widehat{C}(D)$) are satisfied

$$\begin{cases} \varphi_1(P) = \sum\limits_{j=1}^{\infty} \beta_j U_j(P) + \beta_0,\ with\ \sum\limits_{j=1}^{\infty} (\beta_j)^2 < \infty \\ \partial_t u(0, P) = \sum\limits_{j=1}^{\infty} \lambda_j b_j U_j(P) + b_0 = \varphi_1(P) \end{cases} \tag{3.185}$$

Here $\widehat{C}(D) = \mathcal{C}^1(D)$ is consisting from all continuously differentiable functions $\varphi_1(P) \in \mathcal{C}^1(D)$ satisfying

$$(\partial_n \varphi_1)(P \in S) = 0 \ and \ \varphi_1(P) = \sum_{j=1}^{\infty} \beta_j U_j(P) + \beta_0 \, , \ \sum_{j=1}^{\infty} |\beta_j|^2 < \infty \tag{3.186}$$

We get

$$b_0 = \beta_0 \ and \ \lambda_j \, b_j = \beta_j \, , \ j \geqslant 1 \tag{3.187}$$

In conclusion,the mixed hyperbolic problem $(II), (II_a), (II_b)$ has a generalized (weak) solution

$$u(t,.) : [0,T] \to L_2(D)$$

$$u(t,P) = \sum_{j=1}^{\infty} [a_j cos\lambda_j t + b_j sin\lambda_j t] U_j(P) + b_0 t$$

such that

$$U_N(t,P) = \sum_{j=1}^{N} [a_j cos\lambda_j t + b_j sin\lambda_j t] U_j(P) + b_0 t$$

satisfies (II) and (II_a) and (II_b) is fulfilled in a "weak sense"

$$u_N(0,P) = \varphi_0^N(P), \partial_t u_N(0,P) = \varphi_1^N(P)$$

Here the "weak sense" means

$$\lim_{N\to\infty} \varphi_0^N = \varphi_0 \ in \, L_2(D)$$

and

$$\lim_{N\to\infty} \varphi_1^N = \varphi_1 \ in \, L_2(D)$$

3.8.3 Fourier Method, Exercises

Exercise 1

Solve the following mixed problem for a PDE of a parabolic type using Fourier method

$$\begin{cases} (I) \, \frac{1}{a^2} \partial_t u(t,x) = \partial_x^2 u(t,x), \ \ t \in [0,T], \, x \in [0,1] \\ (I_a) \, u(0,x) = \varphi(x), \, x \in [0,1], \ \ \varphi \in \mathcal{C}\prime([0,1];\mathbb{R}) \\ (I_b) \, u(t,0) = u(t,1) = 0, \ \ t \in [0,T] \end{cases}$$

where $\mathcal{C}\prime([0,1];\mathbb{R}) = \{\varphi \in \mathcal{C}([0,1];\mathbb{R}) : \varphi(0) = \varphi(1) = 0\}$.

Hint. We must notice from the very beginning that the space $\mathcal{C}\prime([0,1];\mathbb{R})$ is too large for taking Cauchy condition(I_a) and from the way of solving we are forced to accept only $\varphi \in$ with the following structure

$$\varphi(x) = \sum_{j=1}^{\infty} \alpha_j \widehat{U}_j(x) \ , \ x \in [0,1] \tag{3.188}$$

where

$$\widehat{U}_j(x = \frac{1}{\sqrt{2}} sin\, j\pi\, x,\ j \geqslant 1,\ x \in [0,1],\ \text{(orthogonal in } L_2[0,T]) \tag{3.189}$$

$$\text{satisfy } \int_0^1 \widehat{U}_j(x)\widehat{U}_k(x)dx = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases}$$

and the following series

$$\sum_{j=1}^{\infty} |\alpha_j|^2 < \infty \tag{3.190}$$

is a convergent one. The Fourier method involves solutions as function of the following form

$$u(t,x) = \sum_{j=1}^{\infty} T_j(t)U_j(x) + a_0,\ t \in [0,T],\ x \in [0,1] \tag{3.191}$$

where each term $U_j(t,x) = T_j(t)U_j(x)$ satisfies the parabolic equation (I) and the boundary conditions(I_b). In this respect, from the PDE (I) we get the following

$$\frac{\partial_x^2 U_j(x)}{U_j(x)} = \frac{1}{a^2}\frac{\partial_t T_j(t)}{T_j(t)} = -\mu_j,\ \mu_j \neq 0,\ t \in [0,T],\ x \in [0,1],\ j \geqslant 1 \tag{3.192}$$

which are into a system

$$\begin{cases} \frac{dT_j(t)}{dt} + a^2\mu\, T_j(t) = 0 \ \ t \in [0,T] \\ \frac{d^2U_j}{dx^2}(x) + \mu\, U_j(x) = 0,\ \ x \in [0,1] \end{cases} \tag{3.193}$$

On the other hand, to fulfil the boundary conditions (I_b)we need to impose

$$U_j(0) = U_j(1) = 0 \ ,\ \ j \geqslant 1 \tag{3.194}$$

and to get a solution $\{U_j(x) : x \in [0,1]\}$ satisfying the second order differential equation in (3.193) and the boundary conditions (3.194) we need to make the choice

$$\mu_j = (j\,\pi)^2 > 0,\ U_j(x) = c_j\, sin\, j\pi\, x,\ x \in [0,1],\ j \geqslant 1 \tag{3.195}$$

In addition, we get the general solution $T_j(t)$ satisfying the first equation in (3.193)

$$T_j(t) = a_j[\exp - (j\pi a)^2 t],\ t \in [0,T],\ j \geqslant 1,\ a_j \in \mathbb{R} \tag{3.196}$$

Now the constants c_j in $U_j(x)$ (see (3.195)) are taken such that

$$\widehat{U}_j(x) = \widehat{c}_j\, sin\, j\pi\, x,\ \ x \in [0,1],\ j \geqslant 1 \tag{3.197}$$

is an orthonorma system in $L_2([0,1];\mathbb{R})$ and it can be satisfied noticing that

$$\begin{aligned}\int_0^1 (sin\, j\,\pi x)(sin\, k\,\pi x)dx &= \int_0^1 (cos\, j\,\pi x)(cos\, k\,\pi x)dx \\ &= \int_0^1 [cos(j+k)x]dx \\ &= \int_0^1 (cos\, j\,\pi x)(cos\, k\,\pi x)dx \\ &= \begin{cases} 1\setminus 2 & j=k \\ 0 & j\neq k \end{cases} \end{aligned} \tag{3.198}$$

Here

$$\int_0^1 (cos\, m\,\pi x)dx = 0 \text{ for any } m \geqslant 1$$

and

$$cos\alpha cos\beta = \frac{cos(\alpha+\beta)}{2}$$

are used. From (3.198) we get

$$\widehat{c}_j = \sqrt{2} \;\; j \geqslant 1$$

As a consequence

$$\widehat{U}_j(x) = \sqrt{2}sin\, j\,\pi\, x\,,\; x\in[0,T]\,,\; j\geqslant 1 \tag{3.199}$$

is an orthonormal system in $L_2([0,1];\mathbb{R})$ and the series in (3.191) becomes

$$\widehat{U}(t,x) = \sum_{j=1}^{\infty} a_j \widehat{U}_j(x)[\exp-(j\pi a)^2 t] + a_0 \tag{3.200}$$

Now we are looking for the constants $a_j, j\geqslant 0$ such that the series in (3.200) is uniformly convergent on $(t,x)\in[0,T]\times[0,1]$ and in addition the initial condition(I_a) must be satisfied

$$\widehat{U}(0,x) = \sum_{j=1}^{\infty} a_j \widehat{U}_j(x) + a_0 = \varphi(x) = \widehat{U}(0,x) = \sum_{j=1}^{\infty} \alpha_j \widehat{U}_j(x) \tag{3.201}$$

where $\sum\limits_{j=1}^{\infty} |\alpha_j|^2 < \infty$ is assumed (see(3.190)). As a consequence $a_0 = 0$ and $a_j = \alpha_j\,,\; j\geqslant 1$, are the corresponding Fourier coefficients associated with φ satisfying (3.188) and (3.190).

In conclusion, the uniformly convergent series given in (3.200) has the form

$$\widehat{u}(t,x) = \widehat{U}(t,x) = \sum_{j=1}^{\infty} \alpha_j \widehat{U}_j(x)[\exp-(j\pi\, a)^2 t],\ t \in [0,T], x \in [0,1] \tag{3.202}$$

It is the weak solution of the problem $\{(I),(II_a),(II_b)\}$ in a sense that

$$\widehat{U}_N(t,x) = \sum_{j=1}^{N} \alpha_j \widehat{U}_j(x)[\exp-(j\pi\, a)^2 t] \tag{3.203}$$

satisfies the following properties

$$\lim_{N\to\infty} \widehat{U}_N(t,x) = \widehat{U}(t,x)\ uniformly\ of\ t \in [0,1] \tag{3.204}$$

$$each\ \{\widehat{U}_N(t,x) : (t,x) \in [0,T]\times[0,1]\}\ fulfills\,(I)\,and\,(I_b) \tag{3.205}$$

$$\widehat{U}(0,x) = \varphi(x)\,,\ x \in [0,1]\,,\ ((I_a)\,is\,satisfied\,for\,\widehat{u}) \tag{3.206}$$

provided $\varphi \in \mathcal{C}_0([0,1];\mathbb{R})$ fulfil (3.188) and (3.190).

Exercise 2

Solve the following mixed problem for a PDE of hyperbolic type using Fourier method

$$\begin{cases} (I)\,\partial_t^2 u(t,x) = \partial_x^2 u(t,x),\ \ t \in [0,T],\, x \in [0,1] \\ (I_a)\,u(0,x) = \varphi_0(x),\, \partial_t u(0,x) = \varphi_1(x),\, x \in [0,1],\ \ \varphi_0,\varphi_1 \in \mathcal{C}([0,1];\mathbb{R}) \\ (I_b)\,\partial_x u(t,0) = \partial_x u(t,1) = 0,\ \ t \in [0,T] \end{cases}$$

Hint. As in the previous exercise treating a mixed problem for parabolic equation we must notice that the space of continuous $\mathcal{C}([0,1];\mathbb{R})$ is too large for the initial conditions (II_a) considered here. From the way of solving we are forced to accept only $\varphi_0,\varphi_1 \in \mathcal{C}([0,1];\mathbb{R})$ satisfying the following conditions

$$\varphi_0(x) = \sum_{j=1}^{\infty} \alpha_j V_j(x) + \alpha_0,\ \varphi_1(x) = \sum_{j=1}^{\infty} \beta_j V_j(x) + \beta_0\,,\ x \in [0,1] \tag{3.207}$$

where

$$\{V_j(x) = \sqrt{2} cos\, j\,\pi\, x\,,\ x \in [0,1]\}_{j\geqslant 1} \tag{3.208}$$

is an orthonormal system in $L_2[0,1]$, and the corresponding Fourier coefficients $\{\alpha_j,\beta_j\}_{j\geqslant 1}$ define convergent series

$$\sum_{j=1}^{\infty} |\alpha_j|^2 < \infty\,,\ \sum_{j=1}^{\infty} |\beta_j|^2 < \infty \tag{3.209}$$

The Fourier method involves solutions $u(t,x) : [0,T]\times[0,1] \to \mathbb{R}$ possessing partial derivative $\partial_t u(0,x) : [0,1] \to \mathbb{R}$ and it must be of the following form

$$u(t,x) = \sum_{j=1}^{\infty} T_j(t) V_j(x) + a_0 + b_0 t\,,\ t \in [0,T]\,,\ x \in [0,1] \tag{3.210}$$

Each term $u_j(t,x) = T_j(t)V_j(x)$ must satisfy the hyperbolic equation (II) and the boundary conditions(II_b). It implies the following system of ODE

$$\begin{cases} \frac{d^2T_j(t)}{dt^2} - \mu_j T_j(t) = 0\,,\; t\in[0,T],\; j\geqslant 1 \\ \frac{d^2V_j(t)}{dx^2} - \mu_j V_j(t) = 0\,,\; x\in[0,T],\; j\geqslant 1 \end{cases} \tag{3.211}$$

and, in addition, the boundary conditions

$$\frac{dV_j}{dx}(0) = \frac{dV_j}{dx}(1) = 0\,,\; j\geqslant 1 \tag{3.212}$$

are fulfilled. The conditions (3.212) implies

$$\mu_j = -\lambda_j^2 = -(j\,\pi)^2 \, and \, \{V_j(x) = \sqrt{2} cos\, j\,\pi\, x : x\in[0,1]\}_{j\geqslant 1} \tag{3.213}$$

is an orthonormal system in $L_2[0,1]$. On the other hand, using $\mu_j = -(j\pi)^2$, $j\geqslant 1$ (see(3.213)), from the first equation in (3.211) we get

$$T_j(t) = a_j(cos\, j\,\pi t) + b_j(sin\, j\,\pi t),\;\; t\in[0,T],\; j\geqslant 1 \tag{3.214}$$

Using (3.213) and (3.214), we are looking for $(a_j,b_j)_{j\geqslant 0}$ such that $\{u(t,x) : (t,x) \in [0,t]\times[0,1]\}$ defined in (3.210) is a function satisfying initial condition

$$u(0,x) = \sum_{j=1}^{\infty} a_j V_j(x) + a_0 = \varphi_0 = \sum_{j=1}^{\infty} \alpha_j V_j(x) + \alpha_0,\; x\in[0,1] \tag{3.215}$$

In addition ,the function $\{\partial_t u(t,x) : (t,x)\in[0,T]\times[0,1]\}$ is a continuous one satisfying initial condition

$$\partial_t u(0,x) = \sum_{j=1}^{\infty}(j\,\pi) b_j V_j(x) + b_0 = \varphi_1(x) = \sum_{j=1}^{\infty} \beta_j V_j(x) + \beta_0\,,\; x\in[0,1] \tag{3.216}$$

From the condition (3.215) and (3.216) and assuming (3.209) we get $(a_j,b_j)_{j\geqslant 1}$ of the following form

$$a_j = \alpha_j,\;\; (j\,\pi)b_j = \beta_j\,,\; j\geqslant 1\,, and\; a_0 = \alpha_0\;\, b_0 = \beta_0 \tag{3.217}$$

As a consequence, the series

$$u(t,x) = \sum_{j=1}^{\infty}[a_j(cos\, j\,\pi t) + b_j(sin\, j\,\pi t)]V_j(x) + a_0 + b_0 t \tag{3.218}$$

with the coefficients $(a_j,b_j)_{j\geqslant 1}$ determined in (3.217) as a function for which each term $u(t,x) = a_j(cos\, j\,\pi x) + b_j(sun\, j\,\pi x)V_j(x)$ satisfies the hyperbolic equation (II) and the boundary conditions (II_b). As a consequence, the uniformly convergent series

(3.218) satisfies the mixed problem $(II), (II_a)$ *and* (II_b) in the weak sense, i.e

$$u_N(t,x) = \sum_{j=1}^{N} u_j(t,x) + a_0 + b_0 t, (t,x) \in [0,T] \times [0,1]$$

verifies (II) and (II_b) for each $N \geqslant 1$ and

$$\lim_{N\to\infty} u_N(0,x) = \varphi_0(x),\ \lim_{N\to\infty} \partial_t u_N(0,x) = \varphi_1(x),\ x \in [0,1] \tag{3.219}$$

stand for the initial conditions (II_a).

3.9 Some Nonlinear Elliptic and Parabolic PDE

3.9.1 Nonlinear Parabolic Equation

We consider the following nonlinear parabolic PDE

$$\begin{cases} (\partial_t - \Delta)(u)(t,x) = F(x, u(t,x), \partial_x u(t,x)),\ t \in (0,T],\ x \in \mathbb{R}^n \\ \lim_{t\to 0} u(t,x) = 0\,,\ \ x \in \mathbb{R}^n \end{cases} \tag{3.220}$$

where $\partial_x u(t,x) = (\partial_1 u, ..., \partial_n u)(t,x)$, $\partial_i u = \frac{\partial u}{\partial x_i}$, $\partial_t u = \frac{\partial u}{\partial t}$, $\Delta u = \sum\limits_{i=1}^{n} \partial_i^2 u$. Here $F(x,u,p) : \mathbb{R}^{2n+1} \to \mathbb{R}$ is a continuous function satisfying

$$|F(x,u_0,p_0)| \leqslant C,\ |F(x,u_2,p_2) - F(x,u_1,p_1)| \leqslant L(|u_2 - u_1| + |p_2 - p_1|) \tag{3.221}$$

for any $x \in \mathbb{R}^n$, $|u_i|$, $|p_i| \leqslant \delta$, $i = 0,1,2,$ where $L, C, \delta > 0$ are some fixed constants. A standard solution for (3.220) means a continuous function $u(t,x) : [0,a] \times \mathbb{R}^n \to \mathbb{R}$ which is first order continuously derivable of $t \in (0,a)$ second order continuously derivable with respect to $x = (x_1, ...x_n) \in \mathbb{R}^n$ such that (3.220) is satisfied for any $t \in (0,a), x \in \mathbb{R}^2$ a weak solution for (3.220) means a pair of continuous functions $(u(t,x), \partial_x u(t,x); [0,a] \times R^n \to R^{n+1}$ which are bounded such that the following system of integral equations is satisfied

$$\begin{cases} u(t,x) = \int\limits_0^t [\int\limits_{\mathbb{R}^n} F(y, u(s,y), \partial_y) u(s,y) P(t-s,x,y) dy] ds \\ \partial_x u(t,x) = \int\limits_0^t [\int\limits_{\mathbb{R}^n} F(y, u(s,y), \partial_y u(s,y)) \partial_x P(t-s,x,y) dy] ds \end{cases} \tag{3.222}$$

for any $t \in [0,a], x \in R^n$, where $P(\sigma,x,y)$, $\sigma > 0, x, y \in \mathbb{R}^n$ is the fundamental solution of the parabolic equation $(\partial_\sigma - \Delta_x)P = 0, \sigma > 0, x, y \in \mathbb{R}^n$

$$P(\sigma,x,y) = (4\pi\sigma)^{\frac{-n}{2}} \exp - \frac{|y-x|^2}{4\sigma}, \sigma > 0 \tag{3.223}$$

A direct computation shows that $P(\sigma,x,y)$ satisfy the following properties

$$\int_{\mathbb{R}^n} P(\sigma,x,y)dy = 1,\ \partial_\sigma P(\sigma,x,y)\Delta_x P(\sigma,x,y) \tag{3.224}$$

for any $\sigma > 0,\ x, y \in \mathbb{R}^n$ and

$$\lim_{\sigma\downarrow 0} P(\sigma,x,y) = 0\ if\ x \neq y$$

The unique solution for theintegral equation (3.222) is found using the standard approximations sequence defined recurrently by

$$(u,P)_0(t,x) = (0,0) \in \mathbb{R}^{n+1}\ \ and$$

$$\begin{cases} u_{k+1}(t,x) = \int\limits_0^t [\int\limits_{\mathbb{R}^n} F(y, u_k(s,y), p_k(s,y))P(t-s,x,y)dy]ds \\ p_{k+1}(t,x) = \partial_x u_{k+1}(t,x) = \int\limits_0^t [\int\limits_{\mathbb{R}^n} F(y, u_k(s,y), p_k(s,y))\partial_x P(t-s,x,y)dy]ds \end{cases} \tag{3.225}$$

for any $k \geqslant 0$, $(t,x) \in [0,a] \times \mathbb{R}^n$ where $a > 0$ is sufficiently small such that

$$a\,C \leqslant \delta,\ 2\sqrt{a}\,C\,C_1 \leqslant \delta \tag{3.226}$$

Here the constants $C, \delta > 0$ are given in the hypothesis (3.221) and $C_1 > 0$ fixed satisfies

$$(\pi)^{-n\backslash 2} \int_{\mathbb{R}^n} |z|(\exp - |z^2|)dz \leqslant C_1 \tag{3.227}$$

A constant $a > 0$ verifying (3.226) allows one to get the boundedness of the sequence $\{(u_k,p_k)\}_{k\geqslant 1}$ as in the following lemma

Lemma 3.9.1. *Let $F \in \mathcal{C}(\mathbb{R}^{2n+1}, \mathbb{R})$ be given such that the hypothesis (3.221) is verified. Fix $a > 0$ such that (3.226) are satisfied. Then the sequence $\{(u_k,p_k)\}_{k\geqslant 0}$ of continuous functions constructed in (3.225) has the following properties*

$$|u_k(t,x)| \leqslant \delta,\ |p_k(t,x)| \leqslant \delta,\ \forall\,(t,x) \in [0,a] \times \mathbb{R}^n\,,\ k \geqslant 0 \tag{3.228}$$

$$\begin{aligned} \parallel (u'_{k+1}, p_{k+1}(t) - (u_k,p_k)(t)) \parallel &= \sup_{x\in\mathbb{R}^n} [|u_{k+1}(t,x) - u_k(t,x)| \\ &- |p_{k+1}(t,x) - p_k(t,x)|] \leqslant 2\delta C_a^k (\frac{t^k}{k!})^{1\backslash 3} \end{aligned} \tag{3.229}$$

for any $t \in [0, a]$, $k \geqslant 0$ *where* $C_a = L(C_1\sqrt{a} + a^{2\backslash 3})$.

Remark 3.9.2. *Using the conclusion* (3.229) *of Lemma* (3.9.1) *and Lipschitz continuity of* F *in* (3.221) *we get that the sequence* $\{(u,p)_k\}_{k\geqslant 0}$ *is uniformly convergent of* $(t,x) \in [0,\widehat{a}]\times\mathbb{R}^n$, $\lim\limits_{k\to\infty}(u,p)_k(t,x) = (\widehat{u},\widehat{p})(t,x)$ *there the continuous function*$(\widehat{u},\widehat{p}), t \in [0,\widehat{a}], x \in \mathbb{R}^n$ *satisfies the integral equation* (3.222)*, provided the constant* $\widehat{a} > 0$ *is fixed such that*

$$(\frac{\widehat{a}}{2}^{\frac{1}{3}}).C_{\widehat{a}} = \rho < 1 \tag{3.230}$$

where $C_{\widehat{a}}$ *is given in* (3.229). *In this respect,* $\sum(t,x) = \sum\limits_{k=0}^{\infty}[(u,p)_{k+1} - (u,p)_k](t,x)$ *is bounded by a numerical convergent series* $|\sum(t,x)| \leqslant 2\delta(1+\rho+, ...+\rho^k+...) = \frac{2^\delta}{1-\rho}$ *which alow us to obtain the following*

Lemma 3.9.3. *Let* $F \in \mathcal{C}(\mathbb{R}^{2n+1},\mathbb{R})$ *be given such that the hypothesis* (3.221) *is satisfied. Then there exists a unique solution* $(\widehat{u}(t,x), \widehat{p}tt,x), t \in [0,\widehat{a}], \times \in \mathbb{R}^n$*, verifying the integral equations* (3.222) *and*

$$|\widehat{u}(t,x)|, |\widehat{p}(t,x)| \leqslant \rho(forall) t \in [0,\widehat{a}], \times \in \mathbb{R}^n \tag{3.231}$$

$$\partial_x\widehat{u}(t,x) = \widehat{p}(t,x)\forall t \in [0,\widehat{a}], x \in \mathbb{R}^n \tag{3.232}$$

In addition, $((\widehat{u}(t,x), \partial_x\widehat{u}(t,x)))$ *is the unique weak solution of the nonlinear equation* (3.229)*, i.e*

$$\lim_{\varepsilon\downarrow 0} t(\partial_t - \Delta)\widehat{u_\varepsilon}(t,x) = F(x,\widehat{u}(t,x),\partial_x\widehat{u}(t,x)) \tag{3.233}$$

for each $0 < t \leqslant \widehat{a}, x \in \mathbb{R}^n$ *where*

$$\widehat{u_\varepsilon}(t,x)\int_0^t[\int_{\mathbb{R}^n} F(y,\widehat{u},s,y),\partial_y\widehat{u}(s,y)P(t-s,x,y)dy]ds$$

if $0 < t = \widehat{a}$ *and* $x \in \mathbb{R}^n$ *are fixed.*

3.9.2 Some Nonlinear Elliptic Equations

We consider the following nonlinear elliptic equation

$$\Delta u(x) = f(x,u(x)), x \in R^2, n \geqslant 3 \; where \; f(x,u) : \mathbb{R}^{n+1} \to \mathbb{R} \tag{3.234}$$

is first order continuously differentiable satisfying

$$f(x,u) = 0 for \; x \in \; B(0,b), u \in R, \text{ where } B(0,b) \subseteq \mathbb{R}^n \text{ is fixed} \tag{3.235}$$

$$\lambda = \{\max|\frac{\partial t}{\partial u}|; x \in B(0,b), |u| \leqslant 2K_1\} \tag{3.236}$$

satisfies $\lambda K_0 = \rho \in (0, \frac{1}{2})$ where

$$K_0 = \frac{2b^2}{n-2}, K_1 = C_0 K_0 and C_0 = \{\max|f(x,0)|; x \in B(0,b)\}$$

Lemma 3.9.4. *Assume that $f(x,u); (x,u) \in R^{n+1}$ is given satisfying the hypothesis (3.235) and (3.236). Then there exits a unique bounded solution of the nonlinear equation (3.234) $\widehat{u}(x); R^n \to R$. which satisfies the following integral equation*

$$\widehat{u}(x) = \int\limits_{\mathbb{R}^n} \widehat{f}(y, \widehat{u}(y)|y-x|^{2-n} dy = \int\limits_{B(o,b)} \widehat{f}(y, \widehat{u}(y))|y-x|^{2-n} dy \tag{3.237}$$

where $\widehat{f}(x,u) = \widehat{C} f(x,u)$ and $\widehat{C} = \frac{1}{(2-n)|\Omega_0|}, |\Omega| = meas S(0,1)$.

Remark 3.9.5. *The proof of this result usees the standard computation performed for the Poisson equation in the first section of this chapter. More precisely, let$\Omega_0 = S(0,1) \subseteq \mathbb{R}^n$ be the sphere centered at origin and with radius $\rho = 1$. Denote $|\Omega_0| = meas\Omega_0$ and $\widehat{f}(x,u) = \widehat{C} f(x,u), (x,u) \in \mathbb{R}^n \times \mathbb{R}$ where $\widehat{c} = \frac{1}{(2-n)|\Omega_0|}$. Associate the integral equation (3.237) which can be rewritten as*

$$u(x) = \int\limits_D \widehat{f}(x+z, u(x+z)).|z|^{2-n} dz,\ for x \in B(o,b) \tag{3.238}$$

where$D = B(0,L), L = 2b$. Define a sequence $u_k(x); x \in B(0,b) k \geqslant 0 u_0(x) = 0, x \in B(0,b)$

$$u_{k+1}(x) = \int\limits_D \widehat{f}(x+z, u_k(x+z))|z|^{2-n} dz, k \geqslant 0 \times,\ x \in B(0,b) \tag{3.239}$$

The sequence $\{u_k\}_{k \geqslant 0}$ is bounded and uniformly convergent to a continuous function as following estimates show

$$|u_1(x)| \leqslant \widehat{c} C_0 \int\limits_0^L r dr \int\limits_{\Omega_0} dw = \frac{C_0}{n-2} \frac{L^2}{2} = \frac{C_0}{n-2} 2b^2 = K_1 \tag{3.240}$$

for any $x \in B(0,b)$ and

$$|u_2(x) - u,(x)| \leqslant \int\limits_D 1 + \widehat{f}((x+z),(x+z)) - \widehat{f}(x+z,0)|z|^{2-n} dz \leqslant \tag{3.241}$$

$$\leqslant K_1 max|\frac{\partial \widehat{f}}{\partial u}(y,u)| \int\limits_S |z|^{2-n} dz \leqslant K_1(\lambda.K_0) = K_1 \rho\, for any\, x \in B(0,b)$$

An induction argument leads us to

$$|u_{k+1}(x) - u_k(x)| \leqslant K_1 \rho^k,\ x \in B(0,b),\ k \geqslant 0 \tag{3.242}$$

where $\rho \in (0, 1 \setminus 2)$ *and* $k_1 > 0$ *are defined in* (3.236). *Rewrite*

$$u_{k+1} = u_1(x) + u_2(x) - u_1(x) + \cdots + u_{k+1}(x) - u_k(x) \tag{3.243}$$

and consider the following series of continuous functions

$$\sum(x) = u_(x) + \nu_2(x) + \cdots + \nu_{k+1}(x) + \ldots \tag{3.244}$$

$$x \in B(0, b),\ \nu_{j+1}(x) = u_{j+1}(x) - u_j(x)$$

The series in (3.244) *is dominated by a convergent numerical series*

$$|\sum(x)| \leqslant K_1(1 + \rho + \rho^2 + \cdots + \rho^k + \ldots) = K_1 \frac{1}{1-\rho} \leqslant 2K_1 \tag{3.245}$$

and the sequence defined in (3.239) *is uniformly convergent to a continuous and bounded function*

$$\lim_{k\to\infty} u_k(x) = \widehat{u}(x),\ |\widehat{u}(x)| \leqslant 2K_1,\ x \in B(0, b) \tag{3.246}$$

In addition$\{\widehat{u}(x),\ x \in B(0, b)\}$ *verifies*

$$\widehat{u}(x) \int\limits_{\mathbb{R}^n} \widehat{f}(y, \widehat{u}(y))|y - x|^{2-n} dy = \int\limits_{B(0,b)} \widehat{f}(y, \widehat{u}(y))|y - x|^{2-n} dy \tag{3.247}$$

The solution $\{\widehat{u}(x),\ x \in B(0, b)\}$ *is extended as a continuous function on* $\mathbb{R}^n$ *using the same as integral equation (see* (3.247)*) and the proof of* (3.237) *is complete.*

3.10 Exercises

Weak solutions for parabolic and hyperbolic boundary problems by Fourier's method (P_1). Using Fourier method, solve the following mixed problem for a scalar parabolic equation

$$\begin{cases} (a)\ \partial_t u(t, x) = \partial_x^2 u(t, x)\,,\ t \in [0, T],\ x \in\ [A, B] \\ (b)\ u(0, x) = \varphi_0(x)\,,\ x \in\ [A, B]\,,\ \varphi_0 \in \mathcal{C}([A, B]) \\ (c)\ u(t, A) = u_A,\ u(t, B) = u_B\,,\ t \in [0, T],\ u_A, u_B \in \mathbb{R}\ given \end{cases} \tag{3.248}$$

Hint. Make a function transformation

$$\nu(t, x) = u(t, x) - \{\frac{x - A}{B - A} u_B + \frac{B - x}{B - A} u_A\} \tag{3.249}$$

which preserve the equation (a) but the boundary condition (c) becomes $\nu(t, A) = 0, \nu(t, B) = 0, t \in [0, T]$. The interval $[A, B]$ is shifted into $[0, 1]$ by the

following transformations

$$x = B\,y + (1-y)A,\; w(t,y) = \nu(t, B\,y + (1-y)A),\; y \in [0,1],\; t \in [0,T] \tag{3.250}$$

The new function $\{w(t,y) : y \in [0,T], y \in [0,1]\}$ fulfils a standard mixed problem for a scalar parabolic equation

$$\begin{cases} (a)\ \partial_t\, w(t,y) = \frac{1}{(B-A)^2}\partial_y^2 w(t,y),\ t \in [0,T],\ y \in [0,1] \\ (b)\ w(0,y) = \varphi_1(y) \overset{def}{=} \varphi_0(A + (B-A)y) - \{u_B y + u_A(1-y)\} \\ (c)\ w(t,0) = 0\,,\ w(t,1) = 0\,,\ t \in [0,T] \end{cases} \tag{3.251}$$

$$w(t,y) = \sum_{j=1}^{\infty} T_j(t) W_j(t) \tag{3.252}$$

where

$$\frac{T_j'(t)}{T_j(t)} = \frac{W_j''(y)}{W_j(y)} \frac{1}{(B-A)^2} = -\lambda_j,\ \lambda_j > 0,\ j = 1, 2, ... \tag{3.253}$$

The boundary condition ((3.248), c) must be satisfied by the general solution

$$W_j(y) = C_1\, cos\,(B-A)\sqrt{\lambda_j} y + C_2\, sin\,(B-A)\sqrt{\lambda_j} y \tag{3.254}$$

of (3.253) which implies $C_1 = 0$ and $\lambda_j = (B-A)^{-2}(\pi j)^2$

(P_2). Using Fourier method, solve the following mixed problem for a scalar parabolic equation

$$\begin{cases} (a)\ \partial_t^2 u(t,x) = \partial_x^2 u(t,x)\,,\ t \in [0,T],\ x \in\ [A,B] \\ (b)\ u(0,x) = \varphi_0(x)\,, \partial_t u(0,x) = \varphi_1(x)\,,\ x \in\ [A,B]\,,\ \varphi_0, \varphi_1 \in \mathcal{C}([A,B]) \\ (c)\ u(t,A) = u_A,\ u(t,B) = u_B\,,\ t \in [0,T],\ u_A, u_B \in \mathbb{R}\ given \end{cases}$$

Hint. Make a function transformation

$$\nu(t,x) = u(t,x) - \{\frac{x-A}{B-A}u_B + \frac{B-x}{B-A}u_A\} \tag{3.255}$$

which preserve the hyperbolic equation (a) but the boundary condition (c) becomes

$$\tag{3.256}$$

The interval $[A,B]$ is shifted into $[0,1]$ with preserving the boundary (3.256) if the following transformation are done

$$x = B\,y + (1-y)A,\; w(t,y) = \nu(t, B\,y + (1-y)A),\; y \in [0,1],\; t \in [0,T] \tag{3.257}$$

The new function $\{w(t,y) : y \in [0,T], y \in [0,1]\}$ fulfils a standard mixed problem for a scalar parabolic equation

$$\begin{cases} (a)\ \partial_t^2 w(t,y) = \frac{1}{(B-A)^2}\partial_y^2 w(t,y),\ t \in [0,T],\ y \in [0,1] \\ (b) \begin{cases} w(0,y) = \psi_0(y) \overset{def}{=} \varphi_0(A+(B-A)y) - \{u_B y + u_A(1-y)\},\ y \in [0,1] \\ \partial_t w(0,y) = \psi_1(y) \overset{def}{=} \varphi_1(A+(B-A)y),\ y \in [0,1] \end{cases} \\ (c)\ w(t,0) = 0\,,\ w(t,1) = 0\,,\ t \in [0,T] \end{cases} \tag{3.258}$$

Solution of (3.258) has the form

$$w(t,y) = \sum_{j=1}^{\infty} T_j(t) W_j(t) + C_0 + C_1 t \tag{3.259}$$

and the algorithm of solving repeats the standard computation given (P_1).

3.11 Appendix I Multiple Riemann Integral and Gauss -Ostrogradsky Formula

(A1) We shall recall the definition of the simple Riemann integral $\{f(x) : x \in [a,b]\}$. Denote by Π a partition of the interval $[a,b] \subseteq R$:
$\Pi = \{a = x_0 \leqslant x_1 \leqslant ... \leqslant x_i \leqslant x_{i+1} \leqslant ... \leqslant x_N = b\}$ and for $\Delta x_i = x_{i+1} - x_i, i \in \{0,1,...,N-1\}$, define the norm $d(\Pi) = \max_i \Delta x_i$ of the partition Π. For $\xi_i \in [x_i, x_{i+1}], i \in \{0,1,...,N-1\}$ (marked points of Π) associate the integral sum

$$S_\Pi(f) = \sum_{i=0}^{N-1} f(\xi_i)\Delta x_i$$

Definition 3.11.1. *A number $I(f)$ is called Riemann integral of the function $\{f(x) : x \in [a,b]\}$ on the interval $[a,b]$ if for any $\epsilon > 0$ there is a $\delta > 0$ such that*

$$|S_\Pi(f) - I(f)| < \epsilon \ \ \forall \ \ \Pi \ \ \textit{satisfying} \ \ d(\Pi) < \delta.$$

Remark 3.11.2. *An equivalent definition can be expressed using sequences of partitions $\{\Pi_k\}_{k\geqslant 1}$ for which*

$$\lim_{k\to\infty} S_{\Pi_k}(f) = I(f),$$

where the number $I(f)$ does not depend on the sequence $\{\Pi_k\}_{k\geqslant 1}$ and its marked points. If it is the case we call the number $I(f)$ as the integral of the function $f(x)$ on the interval $[a,b]$ and write

$$\int_a^b f(x)dx = \lim_{k\to\infty} S_{\Pi_k}(f)$$

Finally, there is a third equivalent definition of the integral which uses a "limit following a direction". Let E be consisting of the all partitions Π associated with their marked points, for a fixed $\delta > 0$, denote $E_\delta \subseteq E$ the subset satisfying $d(\Pi) < \delta$, $\Pi \in E_\delta$. The subsets $E_\delta \subseteq E$, for different $\delta > 0$, are directed using $d(\Pi) \to 0$. The integral of the function $\{f(x) : x \in [a,b]\}$ is the limit $J(f)$ of the integral sums following this direction; the three integrals $J(f)$, $\int_a^b f(x)dx$ and $I(f)$ exist and are equal iff one of them exists. It lead us to the conclusion that an integral of a continuous function $\{f(x) : x \in [a,b]\}$ exists.

(A2) Following the above given steps we may and do define the general meaning of a Riemann integral when the interval $[a,b] \subseteq R$ is replaced by some metric space (X,d). Consider that a family of subsets $U \subseteq X$ of X are given verifying the following conditions:

1. The set X and the empty set $\emptyset$ belong to $\mathcal{U}$
2. f $A_1, A_2 \in \mathcal{U}$ then their intersection A_1A_2 belongs to $\mathcal{U}$.
3. If $A_1, A \in U$ and $A_1 \subseteq A$ then there exist $A_2, ..., A_p \in \mathcal{U}$ such that $A = A_1 \cup ... \cup A_p$ and $A_2, ..., A_p \in \mathcal{U}$ are mutually disjoints. A system of subsets $u \subseteq X$ fulfilling (1), (2) and (3) is called demi-ring $\mathcal{U}$. In order to define a Riemann integral on the metric space X associated with a demi-ring $\sqcap$ we need to assume another two conditions.
4. For any $\delta > 0$, there is a partition of the set $X = A_1 \cup ... \cup A_p$, with $A_i \in \mathcal{U}$, $A_iA_j = \phi$ if $i \neq j$ and $d(A_i) = \sup\limits_{(x,y)\in A_i} \rho(x,y) < \delta$, $i \in \{1, ..., p\}$ (This condition remind us the property that X is a precompact metric space). The last condition imposed on the demi-ring $\mathcal{U}$ gives the possibility to measure each individual of U.
5. For each $A \in \mathcal{U}$, there is positive number $m(A) \geqslant 0$ such that $m : \mathcal{U} \to [0, \infty)$ is additive, i.e $m(A) = m(A_1) + m(A_2) + ... + m(A_p)$, if $A = A_1 \cup ... \cup A_p$ and $A_1, ..., A_p$ are mutually disjoints.

The additive mapping $m : U \to [0, \infty)$ satisfying (5) is called measure on the cells composing $\mathcal{U}$. The metric space X associated with a demi-ring of cells U and a finite additive measure $m : u \to [0, \infty)$ satisfying (1)-(5) will be called a measured space $(X, \mathcal{U}, m)$. Let $f(x) : X \to R$ be a real function defined on a measured space $(X, \mathcal{U}, m)$ and for an arbitrary partition $\Pi = \{A_1, ..., A_p\}$ of $x = A_1 \cup ... \cup A_p$ $(A_iA_j = \phi_1 \ i \neq j)$ define an integral sum.

$$S_\Pi(f) = \sum_{i=1}^{\infty} f(\xi_i) m A_i, \; where \, \xi_i \in A_i \; is \, fixed \tag{3.260}$$

The number

$$I_{\sqcap}(f) = \int\limits_X f(x)dx \tag{3.261}$$

is called the integral of the function f on the measured space $(X,\mathcal{U},m)$ if for any $\epsilon > 0$, there is a $\delta > 0$ such that

$$|I_{\sqcap}(f) - S_{\Pi}(f)| < \epsilon \tag{3.262}$$

is verified for any partition Π satisfying $d(\Pi) < \delta$, where $d(\Pi) = \max(d(A_1), ..., d(A_p))$ and $d(A_i) = \sup\limits_{(x,y)\in A_i} \rho(x,y)$. It is easily seen that this definition of an integral on a measured space $(X,\mathcal{U},m)$ coincides with the first definition of the Riemann integral given on a closed interval $[a,b] \subseteq R$.
The other two equivalent definitions using sequence of partitions $\{\pi_k\}_{k\geqslant 1}$, $d(\pi_k) \to 0$, and a "limit following a direction" will replicate the corresponding definitions in the one dimensional case $x \in [a,b]$.
A function $f(x) : \{X,\mathcal{U},m\} \to \mathbb{R}$ defined on a measured space and admitting Riemann integral is called integrable on X , $f \in J(X)$. The following elementary properties of the integral are direct consequence of the definition using sequence of partitions and their integral sums

$$\int_X f(x)dx = c\,m(X),\, if\ f(x) = c(const),\ x \in X \tag{3.263}$$

$$\int_X cf(x)dx = c\int_X f(x)dx,\ f \in J(X)\ and\, c = const \tag{3.264}$$

$$\int_X [f(x) + g(x)]dx = \int_x f(x)dx + \int_X g(x)dx,\ if\, f,g \in J(X) \tag{3.265}$$

$$Any\ f \in J(X)\ is\ bounded\ on X,\ |f(x)| \leqslant c,\ x \in X \tag{3.266}$$

if $f,g \in J(X)$ and $f(x) \leqslant g(x)$, $x \in X$ then

$$\int_X f(x)dx \leqslant \int_X g(x)dx(\int_X f(x)dx \leqslant \int_X |f(x)|dx\, if\ \ f|f| \in J(X)) \tag{3.267}$$

$$C\,m(x) \leqslant \int_X f(x)dx \leqslant C\,m(x),\ if\, f \in J(X)\, and\, c \leqslant f(x) \leqslant C\ \forall\, x \in X\,. \tag{3.268}$$

Theorem 3.11.3. *If a sequence $\{f_k(x) : x \in X\}_{k\geqslant 1} \subseteq J(X)$ converges uniformly on $\{X,\mathcal{U},m\}$ to a function $f(x) : X \to \mathbb{R}$ then $f \in J(X)$ and*

$$\int_X f(x)dx = \lim_{n\to\infty} \int_X f_n(x)dx$$

Hint The proof replicates step by step the standard proof used for $X = [a,b] \subseteq \mathbb{R}$

Example 3.11.1. *(e_1) $X = [a,b] \subseteq \mathbb{R}$ and for a cell of $[a,b]$ can be taken any subinterval containing or not including its boundary points. The measure $m(A) =$*

$\beta-\alpha$, ($A=[\alpha,\beta]$, $A=(\alpha,\beta)$) of a cell is the standard length of it and the corresponding measured space $(X,\mathcal{U},m)$ satisfy the necessary conditions (1)-(5). Notice that the first definition of the integral given in (3.262) *coincides with that definition used in the definition* (3.11.1)
(e_2) Let X be a rectangle in $\mathbb{R}^2$, $X=\{x\in\mathbb{R}^2 : a_1\leqslant x_1\leqslant b_1, a_2\leqslant x_2\leqslant b_2\}$ and as a cell of X we take any subset $A\subseteq X$, $A=\{x\in X : \alpha_1\prec x_1\prec\beta_1,\alpha_2\prec x_2\prec\beta_2\}$ where the sign " $\prec$ " means the standard " $\leqslant$ " or" $<$ " among real numbers.The measure $m(A)$ associated with the cell A is given by its area $m(A)=(\beta_1-\alpha_1).(\beta_2-\alpha_2)$.The necessary condition (1)-(5) are satisfied by a direct inspection and the Riemann integral on X will be denoted by

$$\int_X f(x)dx=\int_{a_1}^{b_1}\int_{a_2}^{b_2} f(x_1,x_2)dx_1\,dx_2\,(\textit{double Riemann integral})$$

Replacing $\mathbb{R}^2$ by $\mathbb{R}^n$, $n\geqslant 2$ and choosing $X=\prod_{i=1}^{n} I_i$ as a direct product of some intervals $I_i=\{x\in\mathbb{R} : a_i\prec x\prec b_i\}$ we define a cell $A=\prod_{i=1}^{n}\{\alpha_i\prec x\prec\beta_i\}$ and its volume $m(A)=\prod_{i=1}^{n}(\beta_i-\alpha_i)$ as the associated measure.In this case the Riemann integral is denoted by

$$\int_X f(x)dx=\int_{a_1}^{b_1}\dots\int_{a_n}^{b_n} f(x_1,\dots,x_n)dx_1\dots dx_n$$

and call it as the n-multiple integral of f.

Theorem 3.11.4. *Let $\{X,\mathcal{U},m\}$ be a measured space and $\mu(x) : X\to\mathbb{R}$ is a uniformly continuous function. Then f is integrable, $f\in J(X)$.*

Proof. By hypothesis the oscillation of the function f on X

$$w_f(X,\delta)=\sup_{\rho(x',x'')\leqslant\delta,x',x''\in X}|f(x')-f(x'')| \tag{3.269}$$

satisfies $w_f(X,\delta)\leqslant\varepsilon$ for some $\varepsilon>0$ arbitrarily fixed provided $\delta>0$ is sufficiently small. In particular,this property is valid on any elementary subset $P\subseteq X$, $P=\bigcup_{i=1}^{p} A_i$, $\{A_1,\dots,A_p\}$ are mutually disjoint cells.Denote $w_f(P,\delta)$ the corresponding oscillation of f restricted to P and notice

$$|S_\pi(f,P)-S'_\pi(f,P)|\leqslant w_f(P,\delta)m(P) \tag{3.270}$$

for any partition π' of P containing the given partition $\pi=\{A_1,\dots,A_p\}$ of the elementary subset $P\subset X(\pi'\supseteq\pi)$.
The property $\pi'\supseteq\pi$ is described by "$\pi'$is following π"(π' is more refine). Using (3.270) for $P=x$ noticing that $w_f(X,\delta)\leqslant\varepsilon$ for $\varepsilon>0$, arbitrarily fixed provided

$\delta > 0$ is sufficiently small, we get that

$$\lim_{\delta(\pi)\to 0} S_\pi(f) = I(f) \ exists$$

as a consequence of the Cauchy criteria applied to integral sums. ☐

Consequence Any continuous function defined on a compact measured space $(X,\mathcal{U},m)$ is integrable(see $f(x) : X \to \mathbb{R}$ is uniformly continuous).

Theorem 3.11.5. *Let $(X,\mathcal{U},m)$ be a measured space and $Z \supseteq X$ is a negligible set.Assume that the bounded function $f(x)$ is uniformly continuous outside of any arbitrary neighborhood of Z,$\mathcal{U}_\sigma(Z) = \{x \in X : \rho(x,Z) < \delta\}$. Then f is integrable on X.*

Proof. By hypothesis Z is a negligible set and for any $\varepsilon > 0$ there is an elementary set $P = \bigcup\limits_{i=1}^{p} A_i$ such that $intP \supseteq Z$ and $m(P) < \varepsilon$. Let $M = \sup\limits_{x\in X} |f(x)|$ and for $\varepsilon > 0$ define

$$P = \bigcup_{i=1}^{p} A_i \ such\ that\ int\,P \supseteq Z\ and\ m(P) \leqslant \frac{\varepsilon}{4M} \tag{3.271}$$

Denote $B = X - P$ and we get $d(Z,B) = 2\rho > 0$ where $d(Z,B) = inf\{d(z,b) : z \in Z, b \in B\}$. By hypothesis,the function f is uniformly continuous outside of a neighborhood $\mathcal{U}_\rho(Z)$,of the negligible set Z, i.e $f(x) : Q \to \mathbb{R}$ is uniformly continuous, where $Q = X \setminus \mathcal{U}_\rho(Z)$. Using Theorem 3.11.4 we get that f restricted to $(Q,\mathcal{U}Q,m)$ is integrable and any integral sum of f on X restricted to P is bounded by ε(see $intP \supseteq Z,\ m(P) \leqslant \frac{\varepsilon}{4M}$).
For an arbitrary partition $\pi = \{C_1,\ldots,C_n\}$ of X we divide it into two classes; the first class contains all cells of π which are included in $P = \bigcup\limits_{i=1}^{p} A_i \supseteq Z_1$ and the second class is composed by the cells of π which have common points with the set $B = X \setminus P$ and are entirely contained in B. In particular, take a partition π with $d(\pi) < \sigma = min(\sigma,\rho)$ where $\sigma > 0$ is sufficiently small such that $|f(x'') - f(x')| < \varepsilon \setminus 2m(X)$ if $\rho(x'',x') < 2\sigma$, $x',x'' \in B$. Let $\pi' \supseteq \pi$ be a following partition(π' is more refined than π). A straight computation allows one to see that

$$|S_\pi(f) - S_{\pi'}(f)| \leqslant |S_\pi(f,P)| + |S_{\pi'}(f,P)| + |S_\pi(f,Q) - S_{\pi'}(f,Q)| \tag{3.272}$$

$Q = X \setminus \mathcal{U}_\rho(Z)$ the first two terms in (3.272) fulfil

$$\begin{cases} |S_\pi(f,P)| \leqslant M \sum\limits_{i=1}^{p} m(A_i) = M\,m(P) \leqslant M.\varepsilon \setminus 4M = \varepsilon \setminus 4 \\ |S_{\pi'}(f,P)| \leqslant M \sum\limits_{i=1}^{p} m(A_i) = M\,m(P) = \varepsilon \setminus 4 \end{cases} \tag{3.273}$$

For the last term in (3.272) we use (3.270) in the proof of Theorem 3.11.5 and $Q =$

$X \setminus \mathcal{U}_\rho(Z) \supseteq X \setminus P = B$. As far as $\sigma > d(\pi) \geqslant d(\pi')$ and $\sigma = \min(\sigma, \rho)$ we obtain

$$\left\{ \begin{array}{l} |S_\pi(f,Q) - S_{\pi'}(f,Q)| \leqslant w_f(Q,\delta)m(X) \leqslant [\varepsilon \setminus 2m(X)]m(X) = \varepsilon \setminus 2 \\ |S_\pi(f) - S_{\pi'}(f)| \leqslant \varepsilon \end{array} \right. \tag{3.274}$$

and the Cauchy criteria used for integral sums $\{S_\pi(f) : d(\pi) \to 0\}$ lead us to the conclusion.The proof is complete. □

Definition 3.11.6. *Let $\{X, \mathcal{U}, m\}$ be a measured space and $G \subseteq X$ a subset. We say that G is a jordanian set with respect to $(\mathcal{U}, m)$ if its boundary $\partial G = \overline{G} \cap \{X \setminus G\}$, is a negligible set. A jordanian closed set A with the property $\overline{int\, A} = A$ is called a jordanian body.*

Remark 3.11.7. *The characteristic function of a jordanian set $G \subseteq X$*

$$\chi_G(x) = \left\{ \begin{array}{ll} 1 & x \in G \\ 0 & x \in X \setminus G \end{array} \right.$$

is an integrable function if the measured space $\{X, \mathcal{U}, m\}$ is compact; $\int_X \chi_G(x)dx = |G|$ is called volume of G.

Definition 3.11.8. *A measured space $\{X, \mathcal{U}, m\}$ for which all cells are jordanian sets is called normally measured;each cell $A \in \mathcal{U}$ has a volume and $m(A) = vol\, A = |A|$*

Let $f(x) : X \to \mathbb{R}$ be a bounded function on a compact measured space $\{X, \mathcal{U}, m\}$ and $G \subset X$ is a jordanian set with the boundary $\Gamma = \partial G$. By definition, the integral of the function f on the set G is given by

$$\int_G f(x)dx = \int_X f(x)\chi_G(x)dx \tag{3.275}$$

If f is a continuous function on G except a negligible Z then $f(x)\chi_G(x)$ is continuous on X except the negligible set $Z \cup \partial G$ and (3.275) exists.

(A_2) Integration and derivation in $\mathbb{R}^n$; Gauss-Ostrogradsky formula

Consider a domain $G \subseteq \mathbb{R}^n$ (a jordanian body)for which the boundary ∂G is piecewise smooth surface. By "piecewise smooth ∂G "we mean that $S = \partial G = \bigsqcup_{p=1}^{q} S_p$ where $(int\, S_i) \cap (int\, S_j) = \phi$ for any $i \neq j \in \{1, \ldots, q\}$ and for each $x \in int\, S_p$, there is a neighborhood $V \subseteq \mathbb{R}^n$ and a first order continuously differentiable mapping

$y = \varphi(u) = \prod_{i=1}^{n-1}(-a_i, a_i) = D_{n-1} \to S_p \cap V$, $\varphi(0) = x$ such that

$rank \parallel \frac{\partial \varphi(u)}{\partial u} \parallel = n-1$. Denote $\varphi = (\varphi_1, ..., \varphi_n)$, $u = (u_1, ..., u_{n-1})$ and define the vectorial product of the vectors $[\frac{\partial \varphi}{\partial u_1}(u), ..., \frac{\partial \varphi}{\partial u_{n-1}}(u)] \subseteq \mathbb{R}^n$ as a vector of $\mathbb{R}^n$ given by the following formula

$$N = det \begin{pmatrix} \overrightarrow{e_1} & . \; . \; . & \overrightarrow{e_n} \\ \frac{\partial \varphi_1}{\partial u_1}(u) & & \frac{\partial \varphi_n}{\partial u_1}(u) \\ . & & . \\ . & & . \\ . & & . \\ \frac{\partial \varphi_1}{\partial u_{n-1}}(u) & . \; . \; . & \frac{\partial \varphi_n}{\partial u_{n-1}}(u) \end{pmatrix} = [\frac{\partial \varphi}{\partial u_1}(u), ..., \frac{\partial \varphi}{\partial u_{n-1}}(u)] \qquad (3.276)$$

where $\{\overrightarrow{e_1}, ..., \overrightarrow{e_n}\} \subseteq \mathbb{R}^n$ is the canonical basis of $\mathbb{R}^n$ and the formal writing of (3.276) stands for a simple rule of computation, when the components of $[\frac{\partial \varphi}{\partial u_1}(u), ..., \frac{\partial \varphi}{\partial u_{n-1}}(u)]$ $\subseteq \mathbb{R}^n$ are involved. In addition, (3.276) allows one to se easily that N is orthogonal to any vector $\frac{\partial \varphi}{\partial u_i}(u) \in \mathbb{R}^n$, $i \in \{1, ..., n-1\}$ and as a consequence N is orthogonal to the point $\varphi(u) \in S_p$. In this respect, the scalar product $< N, \frac{\partial \varphi}{\partial u_i}(u) >$ coincides with the computation of the following expression

$$< N, \frac{\partial \varphi}{\partial u_i}(u) >= det \begin{pmatrix} \frac{\partial \varphi}{\partial u_i}(u) \\ \frac{\partial \varphi}{\partial u_1}(u) \\ . \\ . \\ . \\ \frac{\partial \varphi}{\partial u_{n-1}}(u) \end{pmatrix} = 0 \; for \; each \; i \in \{1, .., n-1\} \qquad (3.277)$$

If it is the case then the computation of the oriented surface integral $\int_S f(x) dS$ on the surface $S = \{S_1, ..., S_q\}$ will be defined by the following formula

$$\int_S f(x) dS = \int_{D_{n-1}} f(\varphi(u)) |N| du \qquad (3.278)$$

where $|N|$ stands for the length of the vector N defined in (3.276). On the other hand, the normalized vector

$$m(u) = N/|N| \qquad (3.279)$$

can be oriented in two opposite directions with respect to the domain G and for the Gauss-Ostrogradsky formula we need to consider that $m(u)$ is oriented outside of the domain G. Rewrite $m(u)$ in ((3.279) as

$$m(u) = \overrightarrow{e_1} cos\, \omega_1 + ... + \overrightarrow{e_n} cos\, \omega_n \qquad (3.280)$$

where ω_i is the angle of the unitary vector $m(u)$ and the axis x_i, $i \in \{1, ..., n\}$. Let $P(x) = P(x_1, ..., x_n)$ be a first order continuously differentiable function in the domain G. Assume that $G \subseteq \mathbb{R}^n$ is simple with respect to each axis x_k, $k \in \{1, ..., n\}$. Then the following formula is valid

$$\int_G \frac{\partial P(x)}{\partial x_k} dx = \oint_S (cos\, \omega_k) P(x) dS \; , \; for\, each \; k \in \{1, ..., n\} \qquad (3.281)$$

where $S = \partial G$ and the surface integral $\oint_S$ is oriented outside of the domain G.

Theorem 3.11.9. *(Gauss(1813)-Ostrogradsky(1828-1834)formula)*

Let $P_k(x) : G \subseteq \mathbb{R}^n \to \mathbb{R}$, be a continuously differentiable function for each $k \in \{1, ..., n\}$ and assume that the jordanian domain G is simple with respect to each axis x_k , $k \in \{1, ..., n\}$. Then the following formula is valid

$$\int_G [\frac{\partial P_1}{\partial x_1}(x) + ... + \frac{\partial P_n}{\partial x_n}(x)]dx = \oint_S [(cos\,\omega_1)P_1(x) + ... + (cos\,\omega_n)P_n(x)]dS \quad (3.282)$$

where the surface integral $\oint_S$ is oriented outside of the domain G and $S = \partial G$

Proof. The meaning that G is simple with respect to each axis x_k , $k \in \{1, ..., n\}$ will be explained for $k = n$ and let $Q \subseteq \mathbb{R}^{n-1}$ be the projection of the domain G on the hyperplane determined by coordinates $x_1, ..., x_{n-1} = \widehat{x}$. It is assumed that $Q \subseteq \mathbb{R}^{n-1}$ is jordanian domain(see jordanian body given in definition 3.11.6 of (A_1))and the jordanian domain $G \subseteq \mathbb{R}^n$ can be described by the following inequalities

$$\varphi(x_1, ..., x_{n-1}) \leqslant x_n \leqslant \psi(x_1, ..., x_{n-1}) \, , \, (x_1, ..., x_{n-1}) = \widehat{x} \in Q \quad (3.283)$$

where $\varphi(\widehat{x}) \leqslant \psi(\widehat{x})$, $\widehat{x} \in Q$ are continuously differentiable functions. The surface $x_n = \psi(\widehat{x})$, $(\widehat{x}, x_n) \in G$ is denoted by S_u(upper surface)and $x_n = \varphi(\widehat{x})$, $(\widehat{x}, x_n) \in G$ will be denoted by S_b(lower surface). Notice that the unitary vector m defined in (3.279) must be oriented outside of G at each point $p \in S_u$. It implies$< m, e_n >\geqslant 0$, similarly $< m, e_n >\leqslant 0$ for each $p \in S_b$. A direct computation of the orthogonal vector N at each point of the surfaces $S_u = \{x_n - \psi(\widehat{x}) = 0\}$ and $S_u = \{x_n - \varphi(\widehat{x}) = 0\}$ will lead us to

$$< m, e_n >=< N \setminus |N|, e_n >= \begin{cases} 1 \setminus (1 + |\partial_{\widehat{x}}\psi(\widehat{x})|^2)^{1\setminus 2} \, , \, x \in S_u \\ 1 \setminus (1 + |\partial_{\widehat{x}}\varphi(\widehat{x})|^2)^{1\setminus 2} \, , \, x \in S_b \end{cases} \quad (3.284)$$

Using the standard decomposition method of a multiple integral into its iterated parts we get

$$\begin{aligned} \int_G \frac{\partial P_n(x)}{\partial x_n} &= \int_Q [\int_{x_n=\varphi(\widehat{x})}^{x_n=\psi(\widehat{x})} \frac{\partial P_n}{\partial x_n}(\widehat{x}, x_n)dx_n]d\widehat{x} \\ &= \int_Q P_n(\widehat{x}, \psi(\widehat{x}))d\widehat{x} - \int_Q P_n(\widehat{x}, \varphi(\widehat{x}))d\widehat{x} \\ &= \int_Q P_n(\widehat{x}, \psi(\widehat{x})) < m, e_n > (1 + |\partial_{\widehat{x}}\psi(\widehat{x})|^2)^{1\setminus 2} d\widehat{x} \\ &\quad + \int_Q P_n(\widehat{x}, \varphi(\widehat{x})) < m, e_n > (1 + |\partial_{\widehat{x}}\varphi(\widehat{x})|^2)^{1\setminus 2} d\widehat{x} \\ &= \int_{S_u} P_n(x) < m, e_n > dS + \int_{S_b} P_n(x) < m, e_n > dS \\ &= \oint_S P_n(x)(cos\,\omega_n)d\,D \end{aligned} \quad (3.285)$$

Here we have used the definition of the unoriented integral given in (3.278) and the proof of (3.282) is complete. □

Remark 3.11.10. *Assuming that* $G = \bigcup_{i=1}^{p} G_i$, *with* $(int\, G_i) \cap (int\, G_j) = \phi$ *if* $i \neq j$ *and each* G_i *is a jordanian domain, simple with respect to any axis* x_k, $k \in \{1, ..., n\}$, *then the Gauss-Ostrogradsky formula* (3.282) *is still valid.*

3.12 Appendix II Variational Method Involving PDE

3.12.1 Introduction

A variational method uses multiple integrals and their extremum values for deriving some PDE as first order necessary conditions. Consider a functional

$$J(z) = \int_{D_m} L(x, z(x), \partial_x z(x))dx\,,\ D_m = \prod_{i=1}^{m}[a_i, b_i] \tag{3.286}$$

where $L(x, z, u) : V \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}\,, V(open) \supseteq D_m$ is a first order continuously differentiable function. We are looking for a continuously differentiable function $\widehat{z}(x) : V \times \mathbb{R}\ \ \widehat{z} \in \mathcal{C}^1(V)$ such that

$$\min J(z) = J(\widehat{z})\ \ z \in \mathcal{A} \tag{3.287}$$

where $\mathcal{A} \subseteq \mathcal{C}^1(V)$ is the admissible set of functions satisfying the following boundary conditions

$$\left\{\begin{array}{l} \left\{\begin{array}{l} z(a_1, x_2, ..., x_m) = z_0^1(x_2, ..., x_m)\,,\ \ (x_2, ..., x_m) \in \prod_{i=1}^{m}[a_i, b_i] \\ z(b_1, x_2, ..., x_m) = z_1^1(x_2, ..., x_m) \end{array}\right. \\ . \\ . \\ . \\ \left\{\begin{array}{l} z(x_1, ..., x_{m-1}, a_m) = z_0^m(x_1, ..., x_{m-1})\,,\ \ (x_1, ..., x_{m-1}) \in \prod_{i=1}^{m-1}[a_i, b_i] \\ z(x_1, ..., x_{m-1}, b_m) = z_1^m(x_1, ..., x_{m-1}) \end{array}\right. \end{array}\right. \tag{3.288}$$

Here the functions z_i^j , $i \in \{0, 1\}$, $j \in \{1, ..., m\}$, describing boundary conditions,are some given continuous functions. This problem belongs to the classical calculus of variations which has a long tradition with significant contributions of Euler(1739)and Lagrange(1736). The admissible class $\mathcal{A} \subseteq \mathcal{C}^1(V)$ is too restrictive and the existence of an optimal solution $\widehat{z} \in \Omega$ is under question even if we assume additional regularity conditions on the Lagrange function L. A more appropriate class of admissible function is defined as follows. Denote $\partial D_m = \Gamma_m$ the boundary of the domain $D_m = \prod_{i=1}^{m-1}[a_i, b_i]$ and let $L_2(D_m; \mathbb{R}^m)$ be Hilbert space of measurable functions $p(x) : D_m \to \mathbb{R}^m$ admitting a finite norm $\parallel p \parallel = (\int_{D_m} |p(x)|^2 dx)^{1\backslash 2} < \infty$. Define the

admissible class $A \subseteq \mathcal{C}(V)$ as follows

$$A = \{z \in \mathcal{C}(D_m) : \partial_x z \in L_2(D_m; \mathbb{R}^m), z|_{\Gamma_m} = z_0\} \tag{3.289}$$

where $z_0 \in \mathcal{C}(\Gamma_m)$ is fixed. In addition,the Euler-Lagrange equation(first order necessary conditions)for the problem

$$\min_{z \in A} J(z) = J(\widehat{z}) \tag{3.290}$$

where J is given in (3.286), and A in (3.289) can be rewritten as a second order PDE provided

$$\text{Lagrange function } L(x, z, u) : V \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R} \tag{3.291}$$

is second order continuously differentiable and $|\partial_u L(x, z, u)| \leqslant \mathcal{C}_N(1 + |u|)\ \forall u \in \mathbb{R}^m$, $x \in D_m$ and $|z| \leqslant N$,where $\mathcal{C}_N > 0$ is a constant for each $N > 0$.

3.12.2 Euler-Lagrange Equation for Distributed Parameters Functionals

A functional of the type (3.286) is defined by a Lagrange function L containing a multidimensional variable $x \in D_m$ (distributed parameter). Assume that L in (3.286) fulfils conditions (3.291) and let $\widehat{z} \in A$ (defined in (3.289)) be the optimal element satisfying (3.290)(locally), i.e there is a ball $B(\widehat{z}, \rho) \subseteq \mathbb{C}(D_m)$ such that

$$J(z) \geqslant J(\widehat{z})\ \forall\ z \in B(\widehat{z}, \rho) \cap A \tag{3.292}$$

Denote

$$W^{1\backslash 2} = \{z \in \mathcal{C}(D_m) : their\ exists\ \partial_x z \in L_2(D_m; \mathbb{R}^m)\} \tag{3.293}$$

and define a linear subspace $Y \subseteq W^{1\backslash 2}$

$$Y = \{y \in W^{1\backslash 2}, y|_{\Gamma_m} = 0\} \tag{3.294}$$

An admissible variation of $\widehat{z}$ is given by

$$z_\varepsilon(x) = \widehat{z}(x) + \varepsilon\, \overline{y}(x)\, ,\ x \in D_m \tag{3.295}$$

where $\varepsilon \in [0, 1]$ and $\overline{y} \in Y$. By definition $z_\varepsilon \in A$, $\varepsilon \in [0, 1]$, and using (3.292) we get the corresponding first order necessary condition of optimality using Frechét differential

$$0 = d\, J(\widehat{z}; \overline{y}) = \lim_{\varepsilon \downarrow 0} \frac{J(z_\varepsilon) - J(\widehat{z})}{\varepsilon}\ ,\ \forall \overline{y} \in Y \tag{3.296}$$

where $d\, J(\widehat{z}; \overline{y})$ is the Frechét differential of J at $\widehat{z}$ computed for the argument $\overline{y}$. The first form of the E-L equation is deduced from (3.296) using the following subspace

$\overline{Y} \subseteq Y$

$$\overline{Y} = sp\{\overline{y} \in \mathcal{C}(D_m) : \overline{y}(x) = \prod_{i=1}^{m} y_i(x_i), y_i \in D_0^1([a_i, b_i])\} \tag{3.297}$$

Here the linear space $D_0^1([\alpha, \beta])$ is consisting of all continuous functions φ which are derivable satisfying $\varphi(\alpha) = \varphi(\beta) = 0$ and its derivative $\{\frac{d\varphi}{dt}, t \in [\alpha, \beta]\}$ is a piecewise continuous function. A direct computation allows us to rewrite

$$0 = d\,J(\widehat{z}, \overline{y})(see\ (11))\ for\ any\ \overline{y} \in \overline{Y} \tag{3.298}$$

as follows

$$0 = \int_{D_m} [\partial_z L(x; \widehat{z}(x); \partial_x \widehat{z}(x))\overline{y}(x) + \sum_{i=1}^{m} \partial_{u_i} L(x; \widehat{z}(x); \partial_x \widehat{z}(x))\partial_{x_i}\overline{y}(x)]dx \tag{3.299}$$

for any $\overline{y} \in \overline{Y}$, where $u = (u_1, ..., u_m)$, $x = (x_1, ..., x_m)$
Denote

$$\begin{cases} \psi_1(x) = \int_{a_1}^{x_1} \partial_z L(t_1, x_2, ..., x_m; \widehat{z}(t_1, x_2, ..., x_m); \partial_x \widehat{z}(t_1, x_2, ..., x_m))dt_1, \\ . \\ . \\ . \\ \psi_m(x) = \int_{a_m}^{x_m} \partial_z L(x_1, x_2, ..., , x_{m-1} t_m; \widehat{z}(tx_1, x_2, ..., x_{m-1}, t_m); \partial_x \widehat{z}(x_1, x_2, ..., x_{m-1}, t_m))dt_m \end{cases} \tag{3.300}$$

and integrating by parts in (3.299) we get (see $\partial_z L = \partial_{x_i}\psi_i$)

$$0 = \int_{D_m} \{\sum_{i=1}^{m} [-\frac{1}{m}\psi_i(x) + \partial_{u_i} L(x; \widehat{z}(x); \partial_x \widehat{z}(x))]\partial_{x_i}\overline{y}(x)\}dx \tag{3.301}$$

for any

$$\overline{y} \in W^{1\backslash 2}, \overline{y}(x) = \prod_{i=1}^{m} y_i(x_i)\,,\ y_i \in D_0^1([a_i, b_i])\ i \in \{1, ..., m\}$$

where $\psi_1(x), ..., \psi_m(x)$ are defined in (3.300). The integral equation (3.301) stands for the first form of the Euler-Lagrange equation associated with the variational problem defined in (3.296). To get a pointwise form of the E-L equation (3.301) we need to assume that

$$\text{The optimal element } \widehat{z} \in \mathcal{C}^2(\theta \subseteq D_m) \tag{3.302}$$

is second order continuously differentiable on some open subset $\theta \subseteq D_m$.

Theorem 3.12.1. *(E-L) Let $L(x, z, u) : D_m \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ be a second order continuously differentiable function and consider that the local optimal element $\widehat{z}$ fulfils* (3.302). *Then the following pointwise $E - L$ equation*

$$\partial_z\, L(x, \widehat{z}(x), \partial_x \widehat{z}(x)) = \sum_{i=1}^{m} \partial_{x_i}[\partial_{u_i}\, L(x, \widehat{z}(x), \partial_x \widehat{z}(x))] \tag{3.303}$$

for any $x \in \theta \subseteq D_m$, is valid and $\widehat{z}|_{\Gamma_m} = z_0 \in \mathcal{C}(\Gamma_m)$

Proof. By hypothesis, the $E-L$ equation (3.301) is verified where $\psi_i(x)$ and $\partial_{u_i}L(x,\widehat{z}(x),\partial_x\widehat{z}(x))$ are first order continuously differentiable functions of $x\in\theta\subseteq D_m$ for any $i\in\{1,...,m\}$. The integral in (3.301) can be rewritten as follows

$$\sum_{i=1}^{m}[-\frac{1}{m}\psi_i(x)+\partial_{u_i}L(x;\widehat{z}(x);\partial_x\widehat{z}(x))]\partial_{x_i}\overline{y}(x)=E_1(x)-E_2(x) \tag{3.304}$$

$$=\sum_{i=1}^{m}\partial_{x_i}\{[-\frac{1}{m}\psi_i(x)+\partial_{u_i}L(x;\widehat{z}(x);\partial_x\widehat{z}(x))]\overline{y}(x)\}-$$

$$-\sum_{i=1}^{m}\{\partial_{x_i}[-\frac{1}{m}\psi_i(x)+\partial_{u_i}L(x;\widehat{z}(x);\partial_x\widehat{z}(x))]\}\overline{y}(x)$$

for any $x\in\theta\subseteq D_m$. For each $x_0\in\theta$ arbitrarily fixed, let $D_{x_0}\subseteq\theta$ be a cube centered at x_0 and using (3.304) for

$$\overline{y}\in\overline{Y},\ \overline{y}(x)=\prod_{i}^{m}y_i(x_i),\ \overline{y}\in\mathcal{C}(D_{x_0}),\ \overline{y}|_{\partial D_{x_0}}=0 \tag{3.305}$$

we rewrite (3.301) restricted to the cube $D_{x_0}\subseteq\theta$

$$\int_{D_{x_0}}E_1(x)Dx=\int_{d_{x_0}}E_2(x)dx \tag{3.306}$$

On the other hand,applying Gauss-Ostrogradsky formula to the first integral in (3.306) we get

$$\int_{D_{x_0}}E_1(x)dx=\oint_{\partial D_{x_0}}\overline{y}(x)\{\sum_{i=1}^{m}[-\frac{1}{m}\psi_i(x)+\partial_{u_i}L(x;\widehat{z}(x),\partial_x\widehat{z}(x))]cos\,\omega_i\}dS \tag{3.307}$$

and using (3.305) we obtain

$$0=\int_{D_{x_0}}E_1(x)dx=\int_{d_{x_0}}E_2(x)dx \tag{3.308}$$

for any $\overline{y}\in\overline{Y}$ satisfying (3.305) where

$$\begin{aligned}E_2(x)&=\{\sum_{i=1}^{m}[-\frac{1}{m}\psi_i(x)+\partial_{u_i}L(x;\widehat{z}(x),\partial_x\widehat{z}(x))]\}\overline{y}(x)\\&=\{-\partial_zL(x;\widehat{z}(x),\partial_x\widehat{z}(x))\\&+\sum_{i=1}^{m}\partial_{u_i}L(x;\widehat{z}(x),\partial_x\widehat{z}(x))\}\overline{y}(x)\end{aligned} \tag{3.309}$$

Assimilating (3.308) as an equation for a linear functional on the space $\overline{Y}_{x_0}=\{\overline{y}\in$

$\mathcal{C}(D_{x_0}) : \overline{y}|_{\partial D_{x_0}} = 0\}$

$$0 = \int_{D_{x_0}} E_1(x)dx = \int_{D_{x_0}} h(x)\overline{y}(x)dx, \ \forall \overline{y} \in \overline{Y}_{x_0} \tag{3.310}$$

a standard argument used in the scalar case can be applied here and it shows that

$$h(x) = 0 \ \forall \ x \in D_{x_0} \tag{3.311}$$

where

$$h(x) = \sum_{i=1}^{m} \partial_{x_i}[\partial_{u_i} L(x; \widehat{z}(x), \partial_x \widehat{z}(x))] - \partial_z L(x; \widehat{z}(x), \partial_x \widehat{z}(x))$$

In particular $h(x_0) = 0$ where $x_0 \in \theta$ is arbitrarily fixed and the proof is complete. □

Remark 3.12.2. *The equation* (3.310) *is contradicted if we assume that* $h(x) > 0$.*It is acomplished by constructing an auxiliary function* $\overline{y}_0(x) = \sigma^2 - |x - x_0|^2$ *on the ball* $x \in B(x_0, \sigma) \subseteq D_{x_0}$ *which satisfy*

1. $\overline{y}_0(x) = 0, \ \forall \ x \in \Gamma_{x_0} = boundary\ of\ B(x_0, \sigma)$
2. $\overline{y}_0(x) > 0, \ \forall \ x \in int\, B(x_0, \sigma)$
3. $h(x) > 0, \ \forall \ x \in int\, B(x_0, \sigma)\, if \ \sigma > 0$ *is sufficiently small*

Define $\overline{y}_0(x) = 0$ *for any* $x \in D_{x_0} \setminus B(x_0, \sigma)$ *and*

$$\int_{D_{x_0}} h(x)\overline{y}_0(x)dx > 0 \quad contradicting$$

.

3.12.3 Examples of PDE Involving E-L Equation

(E_1) Elliptic Equations

Example 3.12.1. *Consider Laplace equation*

$$\Delta z(x) = \sum_{i=1}^{m} \partial_i^2 z(x) = 0$$

on a bounded domain $x \in D_m^0 = \prod_{i=1}^{m}(a_i, b_i), D_m = \sum_{i=1}^{m}[a_i, b_i]$*, associated with a Drichlet boundary condition* $z \setminus \partial D_m = z_0(x)$ *where* $z_0 \in \mathcal{C}(\partial D_m)$*.*
Define Dirichlet integral

$$D(z) = \frac{1}{2}\int_{D_m} |\partial_x z(x)|^2 dx\,,\; \partial_x z(x) = (\partial_1 z(z), ..., \partial_m z(x)),$$

and notice that the corresponding Lagrange function is given by

$$L(x, z, u) = \frac{1}{2}|u|^2\,,\; u \in \mathbb{R}^m$$

If it is the case then compute $\partial_z L = 0$ *and* $\sum_{i=1}^{m} \partial_{x_i}[\partial_{u_i} L(x; \widehat{z}(x), \partial_x \widehat{z}(x))] = \Delta \widehat{z}(x)$ *which allows one to see that (E-L) equation* (3.303) *coincides with the above given Dirichlet problem provided* $\{\widehat{z}(x) : x \in int\, D_m\}$ *is second order continuously differentiable*

Example 3.12.2. *Consider Piosson rquation* $\Delta z(x) = f(x0\,,\; x \in \overset{0}{D}_m$ *associated with a boundary condition* $z|_{\partial D_m} = z_0$*, where* $z_0 \in \mathcal{C}(\partial D_m)$*.*
This Dirichlet problem for Poisson equation can be deduced from (E-L) equation (3.303)*(see theorem (E-L))and in this respect associated the following functional*

$$J(z) = \int_{D_m} \{1 \setminus 2|\partial_x z(x)|^2 + f(x)z(x)\}dx$$

where $L(x, z, u) = 1 \setminus 2|u|^2 + f(x)z\,,\; u \in \mathbb{R}^m,\; z \in \mathbb{R}$*, is the corresponding Lagrange function.Notice that* $\partial_z L = f(x)$ *and*

$$\sum_{i=1}^{m} \partial_{x_i}[\partial_{u_i} L(x; \widehat{z}(x), \partial_x \widehat{z}(x))] = \Delta \widehat{z}(x)$$

allows one to write (E-L) equation (3.303) *as*

$$\Delta \widehat{z}(x) = f(x),\quad x \in \overset{0}{D}_m\;\; \widehat{z}|_{\partial D_m} = z_)(x),\;\; for\;\; z_0 \in \mathcal{C}(\partial D_m)\,.$$

Example 3.12.3. *A semilinear Poisson equation* $\Delta z(x) = f(x),\quad x \in \overset{0}{D}_m$*, associated with a Dirichlet boundary condition* $z|_{\partial D_m} = z_)(x),\;\; for\;\; z_0 \in \mathcal{C}(\partial D_m)$ *can be deduced from (E-L)equation* (3.303) *provided we associate the following functional*

$$J(z) = \int_{D_m} \{1 \setminus 2|\partial_x z(x)|^2 + g(z(x))\}dx$$

where $g(z) : \mathbb{R} \to \mathbb{R}$ *is a primitive of* $f(z)\,,\; \frac{dg|z|}{dz} = f(z)\,,\; z \in \mathbb{R}$
Similarly, the standard argument used in examples 3.12.1 and 3.12.2 lead us to the

following non linear elliptic equation

$$-\Delta\widehat{z}(x) = \widehat{z}(x)|\widehat{z}(x)|^{p-1} + f(\widehat{z}(x))\,,\ \ x \in \overset{0}{D}_m$$

provided $L(x,z,u) = \{\frac{1}{2}|u|^2 - \frac{1}{p+1}|z|^{p+1} - g(z)\}\,,\ p \geqslant 1$ *where* $g(z) : \mathbb{R} \to \mathbb{R}$ *is a primitive of* $f(z)$*. Assuming that* $f(z)$ *satisfy* $f(0) = 0,\ \lim\limits_{z\to\infty} \frac{f(z)}{|z|^p} = 0,\ for\ p = 3$ *we take* $g(z) = \frac{1}{2}\lambda|z|^2$ *and the corresponding (E-L) equation coincides with so called Yang-Milles equation which is significant in Physics.*

(E_2) **Wave equation**

(1) Consider $(x,t) \in \mathbb{R}^3 \times \mathbb{R} = \mathbb{R}^4$ and a bounded interval $D_4 \subseteq \mathbb{R}^4$. Associated the following functional

$$J(z) = \int_{D_4} 1 \setminus 2\{\partial_t^2|\partial_x z(t,x)|^2\}dxdt$$

Notice that the corresponding (E-L) equation (3.303) can be written as a wave equation

$$\Box\widehat{z}(t,x) = \partial_t^2\widehat{z}(t,x) - \Delta_x\widehat{z}(t,x) = 0,\ (t,x) \in \overset{0}{D}_4$$

("$\Box$" d'Alembert operator)

(2) With the same notations as above we get Klein-Gordan equation(mentioned in math.Physics equation)

$$\Box\widehat{z}(t,x) + k^2\widehat{z}(t,x) = 0\,,\ \ (t,x) \in \overset{0}{D}_4$$

which agrees with the following Lagrange Function

$$L(x,z,u) = \frac{1}{2}\{(u_0)^2 - \sum_{i=1}^{3}(u_i)^2 - k^2\,z^2\},\ u = (u_0,u_1,u_2,u_3)$$

Adding $\frac{\lambda}{4}z^4 + j\,z$ to the above L we get another Klein-Gordon equation

$$\Box\widehat{z}(t,x) + k^2\widehat{z}(t,x) = \lambda(\widehat{z}(t,x))^3 + j,\ (t,x) \in \overset{0}{D}_4$$

(E_3) PDE involving mimimal-area surface

Looking for a minimal-area surface $z = \widehat{z}(x), x \in D_m$ satisfying $\widehat{z}|_{\partial D_m} = z_0 \in \mathcal{C}(\partial D_m)$ we associate the functional

$$J(z) = \int_{D_m} \sqrt{1 + |\partial_x z(x)|^2}dx$$

The corresponding (E-L) equation (3.303) is $\mathrm{div}(T(z))(x) = 0\,,\ x \in \overset{0}{D}_4$ where $T(z)(x) = \partial_x z(x) \setminus \sqrt{1 + |\partial_x z(x)|^2}$

(E_4) ODE as (E-L)equation for m=1
A functional

$$J(y) = \int_a^b \sqrt{1 + (y'(x))^2} dx$$

stands for the length of the curve $\{y(x) : x \in [a, b]\}$ and the corresponding (E-L) equation (3.303) is

$$\begin{cases} \frac{d}{dx} \frac{\widehat{y}'(x)}{\sqrt{1+(\widehat{y}'(x))}} = 0\,, \ x \in (a, b) \\ \widehat{y}(a) = y_a,\, \widehat{y}(b) = y_b \end{cases}$$

The solutions are expressed by linear $\widehat{y}(x) = \alpha\, x + \beta,\ \ x \in \mathbb{R}$, where $\alpha, \beta \in \mathbb{R}$ are determined such that the boundary conditions $\widehat{y}(a) = y_a$, $\widehat{y}(b) = y_b$ are satisfied.

3.13 Appendix III Harmonic Functions; Recovering a Harmonic Function from its Boundary Values

3.13.1 Harmonic Functions

A vector field $H(x) = (H_1(x), \ldots, H_n(x)) : V \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is called harmonic in a domain $V \subseteq \mathbb{R}^n$ if $H \in \mathcal{C}^1(V; \mathbb{R}^n)$ and

$$\mathrm{div} H(x) = \sum_{i=1}^{n} \partial x_i H_i(x) = 0, (\partial x_i H_j - \partial x_j H_i)(x) = 0, i, j \in \{i, \ldots, n\} \qquad (3.312)$$

for any $x \in V$. In what follows we restrict ourselves to simple convex domain V and notice that the second condition in (3.312) implies that $H(x) = grad\phi(x) = \partial_x \phi(x)$ of some scalar function ϕ (H has a potential) where ϕ is second order continuously differentiable. Under these conditions, the first constraint in (3.312) can be viewed as an equation for the scalar function ϕ

$$0 = \mathrm{div} H(x) = \mathrm{div}(\mathrm{grad}\phi)(x) = \sum_{i=1}^{n} \partial_i^2 \phi(x) = \Delta\phi(x) \qquad (3.313)$$

Any scalar function $\phi(x) : V \subseteq \mathbb{R}^n \to \mathbb{R}$ which is second order continuously differentiable and satisfies the Laplace equation (3.313) for $x \in V$ will be called a harmonic function on V.

3.13.2 Green Formulas

Let $V \subseteq \mathbb{R}^n$ be a bounded domain with a piecewise smooth boundary $S = \partial V$; consider a continuously derivable scalar field $\psi(x) : V \to \mathbb{R}^n$. By a direct computation we get $(\partial = (\partial_1, \ldots, \partial_n))$

$$\mathrm{div}\psi R =< \partial, \psi R >= \psi < \partial, R > + < R, \partial\psi >= \psi\Delta\phi + < \partial\phi, \partial\psi > \qquad (3.314)$$

Applying Gauss-Ostrigradsky formula and integrating both terms of equality we get

$$\begin{aligned} \int_V \psi\Delta\phi dx + \int_V < \partial\phi, \partial\psi > dx &= \int_V (\mathrm{div}\psi R) dx \\ &= \oint_V < m, \psi R > dS \\ &= \oint_S \psi < m, R > dS \\ &= \oint_S \psi(D_m\phi) dS \end{aligned}$$

which stands for

$$\int_V < \partial\phi, \partial\psi > dx + \int_V \psi\Delta\phi dx = \oint_S \psi(D_m\phi) dS \qquad (3.315)$$

Here $D_m\phi =< m, \partial\phi >$ is the derivative of the scalar function ϕ in the normal direction represented by the unitary orthogonal vector m at the surface $S = \partial V$ oriented outside of V. The expression in 3.315 is the first Green formula and by permutation and subtracting we get the second Green formula

$$\int_V (\psi\Delta\phi - \phi\Delta\psi) dx = \oint_S [\psi(D_m\phi) - \phi(D_m\psi)] dS \qquad (3.316)$$

Theorem 3.13.1. **(a)** *If a harmonic function h vanishes on the boundary $S = \partial V$ then $h(x) = 0$ for any $x \in intV$*

(b) *If h_1 and h_2 are two harmonic functions satisfying $h_1(x) = h_2(x)$, $x \in S = \partial V$, then $h_1(x) = h_2(x)$ for all $x \in int\,V$.*

(c) *If a harmonic vector field $H(x)$ satisfies $< m, H >= 0$ on the boundary $S = \partial V$ then $H(x) = 0$ for any $x \in int\,V$.*

(d) *If H_1 and H_2 are two harmonic vector fields satisfying $< m, H_1 >=< m, H_2 >$ on the boundary $S = \partial V$ then $H_1(x) = H_2(x)$ for any $x \in int\,V$.*

Proof. (a): Using the first Green formula 3.315 for $\phi = \psi = h$ and $\Delta h = 0$ we get $\int_V |\partial h|^2 dx = 0$ and therefore $\partial h = gradh(x) = 0$ for any $x \in V$ which implies $h(x) = const$, $x \in V$. Using $h(x) = 0$, $x \in S = \partial V$. We conclude $h(x) = 0$ for any

$x \in V$.
(b): For $h(x) = h_1(x) - h_2(x)$ use the conclusion of (a).
(c): By hypothesis $H(x) = \partial h(x)$, $x \in V$, where the potential function $h(x)$, $x \in V$, is harmonic, $\Delta h = 0$. Using the first Green formula (3.315) for $\phi = \psi = h$ with $\Delta\phi = 0$ and $D_m\phi =< m, H > (x) = 0$ for $x \in S = \partial V$.. We get

$$\int_V |\partial h|^2 dx = \int_V |H(x)|^2 dx = 0 \text{ and } H(x) = 0 \text{ for all} x \in V.$$

(d): Follows from (c) using the same argument as in (b). The proof is complete. □

Theorem 3.13.2. **(a)** *If $h(x)$ is harmonic on the domain $V \subset \mathbb{R}^n$ then* $\oint_S (D_m h)(x) dS = 0$, *where* $D_m h =< m, \partial h >$.

(b) *Let $B(y,r) \subset V$ be a ball centered at $y \in intV$, and h is a harmonic function on $V \subset \mathbb{R}^n$. Then the arithmetic mean value on $\sum = \partial B(y,r)$ equals the value $h(y)$ at the center y,*

$$h(y) = \frac{1}{|\sum|} \oint_{\sum} h(x) dS.$$

(c) *A harmonic function $h(x) : \mathbb{R}^n \to \mathbb{R}$ which satisfies $\lim_{|x|\to\infty} h(x) = 0$ is vanishing everywhere, $h(x) = 0$, for all $x \in \mathbb{R}^n$*

Proof. (a): Use (3.316) for $\phi = h$, $\psi = 1$.
(b): Let $B(y,\rho) \subset B(y,r)$ be another ball, $\rho < r$, and denote $\sum_r = \partial B(y,\rho)$ the corresponding boundaries. Define $V_{r\rho} = W - Q$ and notice that the boundary $S = \partial V_{r\rho} = \sum_r \coprod \sum_\rho$, where $W = B(y,r)$, $Q = B(y,\rho)$. The normal derivative $D_m =< m, \partial >$ is oriented outside of the domain $V_{r\rho}$ and it implies

$$D_m = \partial_r \text{ on } \sum_r$$

$$D_m = -\partial_r \text{ on } \sum_\rho$$

Take $V = V_{r\rho}$, $\varphi = h$, $\psi = \frac{1}{|x-y|^{n-2}}$ in the second Green formula ((3.316)). It is known that $\psi(x)$, $x \in V$, is a harmonic function on V (see $x \neq y$) and as a consequence the left hand side in ((3.316)) vanishes. The corresponding right hand side in ((3.316)) can be written as a difference on $\sum_r$ and $\sum_\rho$ and the equation ((3.316))lead us to

$$\oint_{\sum_r} [\frac{1}{r^{n-2}}\frac{\partial h}{\partial r} - h\frac{\partial}{\partial r}(\frac{1}{r^{n-2}})] dS = \oint_{\sum_\rho} [\frac{1}{\rho^{n-2}}\frac{\partial h}{\partial r} - h\frac{\partial}{\partial r}(\frac{1}{r^{n-2}})] dS.$$

Performing the elementary derivatives we get

$$\frac{1}{r^{n-2}} \oint_{\sum_r} \frac{\partial h}{\partial r} dS + \frac{n-2}{r^{n-1}} \oint_{\sum_r} h\, dS = \frac{1}{\rho^{n-2}} \oint_{\sum_\rho} \frac{\partial h}{\partial r} dS + \frac{n-2}{\rho^{n-1}} \oint_{\sum_\rho} h\, dS.$$

The first term in both sides of the last equality vanishes (see (a)). Let $|S_1|$ be the area of the sphere $S(0,1) \subseteq \mathbb{R}^n$ and dividing by $|S_1|$. We get $\frac{1}{r^{n-1}|S_1|}\oint_{\sum_r} h\,dS = \frac{1}{\rho^{n-1}|S_1|}\oint_{\sum_\rho} h\,dS$, where $r^{n-1}|S_1| = |\sum_r|$, $\rho^{n-1}|S_1| = |\sum_\rho|$. Using the continuity property of h and letting $\rho \to 0$ from the last equality we get the conclusion (b). (c) : Using the conclusion of (b) for $r \to \infty$, we get conclusion (c). $\square$

3.13.3 Recovering a Harmonic Function Inside a Ball I

By Using its Boundary Values

The arguments for this conclusion are based on the second Green formula written on a domain $V = W - Q$, where $W = B(0,r)$, $Q = B(y,\rho) \subseteq B(0,r)$. This time, we use the following harmonic functions on $V(\partial V = S = \sum_r \coprod \sum_\rho)$

$$\varphi = h(x),\ \psi = \psi(x) = \frac{1}{|x-y|^{n-2}} - \frac{r^{n-2}}{|y|^{n-2}}\frac{1}{|x-y*|^{n-2}} \tag{3.317}$$

where $y* = \frac{r^2}{|y|^2}\,y$, $\psi(x) = 0$, $x \in \sum_r = \partial B(0,r)$ and $\psi_0(x) = \frac{r^{n-2}}{|y|^{n-2}}\frac{1}{|x-y*|^{n-2}}$ is harmonic on W. The second Green formula becomes

$$-\oint_{\sum_r} h(x)\frac{\partial\psi}{\partial r}dS = \oint_{\sum_\rho}[\psi\frac{\partial h}{\partial\rho}(x) - h\frac{\partial\psi}{\partial\rho}(x)]dS \tag{3.318}$$

where $\frac{\partial}{\partial\rho}$ stands for the derivative following the direction of the radius $[y,x]$, $x \in \sum_\rho$. A direct computation lead us to

$$\begin{cases} |\oint_{\sum_\rho}\psi\frac{\partial h}{\partial\rho}(x)dS| \leqslant c_1\frac{1}{\rho^{n-2}}c_2\rho^{n-1} \to 0\,\text{for}\,\rho\to 0 \\ |\oint_{\sum_\rho} h\frac{\partial\psi_0(x)}{\partial\rho}dS| \leqslant c\rho^{n-1} \to 0\,\text{for}\,\rho\to 0 \end{cases} \tag{3.319}$$

and

$$\oint_{\sum_\rho} h\frac{\partial}{\partial\rho}\frac{1}{\rho^{n-2}}ds = (n-2)\oint_{\sum_\rho} h\frac{1}{\rho^{n-1}}ds = \frac{(n-2)\mid S_1\mid}{\mid\sum_\rho\mid} \tag{3.320}$$

$$\oint_{\sum_\rho} h\,ds \longrightarrow (n-2)\mid S_1\mid h(y),\ for \rho \longrightarrow 0$$

Letting $\rho \longrightarrow 0$ from (3.318) we get

$$h(y) = \frac{-1}{(n-2)\mid S_1\mid}\oint_{\Sigma} h(x)\frac{\partial\psi}{\partial r}(x)\,ds \tag{3.321}$$

It remains to compute $\frac{\partial\psi(x)}{\partial r}$ on $x \in \sum_r$ and it is easily seen that $see\, \psi(x) = const\, x \in \sum_r$

$$\frac{\partial\psi}{\partial r}(x) = -\mid grad\,\psi(x) \mid \tag{3.322}$$

By definition, y^* is taken such that

$$\frac{\mid x-y\mid^2}{\mid x-y^*\mid^2} = \frac{r^2 - 2 < x,y > + \mid y\mid^2}{r^2 - 2 < x,y > \frac{r^2}{|y|^2} + \frac{r^4}{|y|^2}} = \frac{\mid y\mid^2}{r^2} = const \tag{3.323}$$

for any $x \in \sum_r$ and the direct computation of $\mid grad\psi(x) \mid$ shows that

$$\mid grad\,\psi(x) \mid = \frac{(n-2)(r^2 - \mid y\mid^2)}{r \mid x-y\mid^n}\, , \mid x \mid = r \tag{3.324}$$

provided (3.323) is used. Using (3.322) and (3.324) int (3.321) we get the following Poisson formula

$$h(y) = \frac{r^2 - \mid y\mid^2}{\mid S_1 \mid r} \oint\limits_{\Sigma} \frac{h(x)}{\mid x-y\mid^n}\, ds = \oint\limits_{\Sigma} P(x,y)\, h(x)\, ds, \tag{3.325}$$

where Poisson kernel

$$P(x,y) = \frac{r^2 - \mid y\mid^2}{\mid S_1 \mid r} \frac{1}{\mid x-y\mid^n} > 0 \ for\ any\ \ y \in\ B(0,r) \tag{3.326}$$

Theorem 3.13.3. *Let $\lambda(x) : \Sigma \subseteq \mathbb{R}^n \to \mathbb{R}$ be a continuous function where $\Sigma = \partial W$ is the boundary of a fixed ball $W \subset \mathbb{R}^n$. Then there exists a unique continuous function $h(y) : W \longrightarrow R$ which is harmonic for $y \in\ int\, W$ and coincides with$\lambda(x)$ on the boundary Σ.*

Proof. Let $r > 0$ be the radius of the ball$W$ and for any $y \in\ int\, W$ define $h(y)$ as follows.

$$h(y) = \oint\limits_{\Sigma} P(x,y)\lambda(x) ds \tag{3.327}$$

where $P(x,y) = \frac{r^2 - |y|^2}{|S_1| r} \frac{1}{|x-y|^n}$ is the Poisson kernel $(see(3.326))$ We know that both $\varphi(y) = \frac{1}{|x-y|^{n-2}}$, $y \in\ int\, W$ and its derivatives $\frac{\partial\varphi(y)}{\varphi y_i}$, $y \in\ intW, i \in 1, 2 \ldots, n$ are harmonic function. As a consequence the following linear combination

$$
\begin{aligned}
\varphi(y) - \frac{2}{n-2}\sum_{i=1}^{n} x_i \frac{\partial\varphi(y)}{\partial y_i} &= \frac{1}{\mid x-y\mid^{n-2}} + 2\sum_{i=1}^{n}\frac{x_i(y_i-x_i)}{\mid x-y\mid^{n}} \\
&= \frac{1}{\mid x-y\mid^{n}}(\sum_{i=1}^{n}(x_i-y_i)^2 + 2\sum_{i=1}^{n}x_iy_i - 2\sum_{i=1}^{n}x_i^2) \\
&= \frac{1}{\mid x-y\mid^{n}}\sum_{i=1}^{n}(y_i^2-x_i^2) \\
&= \frac{\mid y^2\mid -r^2}{\mid x-y\mid^{n}} = -\mid S_i\mid P(x,y) \qquad (3.328)
\end{aligned}
$$

is a harmonic function. In conclusion, $h(y)$, $y \in\ int\, W$, defined in ((3.327)is a harmonic function. On the other hand, for a sequence $\{y_m\}_{m\geqslant 1} \subset\ intW$, $\underset{m\longrightarrow\infty}{lim} y_m = \chi_0 \in \Sigma$. We get

$$\oint_{\Sigma} P(x,y_m)dS = 1(see(3.325), forh \equiv 1) \qquad (3.329)$$

$$\lim_{m\longrightarrow\infty}\int_{\Sigma} P(x,y_m)dS = 0, where\Sigma' = \partial B(x_0,\delta)\cap\Sigma, \delta > 0, \qquad (3.330)$$

and $\mid x-y_m\mid\geqslant c>0, m\geqslant 1$, are used. $\square$

Using (3.329) and (3.330) we see easily that

$$\lim_{m\to\infty}\oint_{\Sigma} P(x,y_m)\lambda(x)ds = \lambda(x_0) \qquad (3.331)$$

and $\{h(y) : y \in W\}$ is a continuous function satisfying

$$h(x) = \lambda(x), x \in \sum \qquad (3.332)$$

$$\triangle h(y) = 0, (forall) y \in intW \qquad (3.333)$$

A continuous function satisfying (3.332) and (3.333) is unique provided the maximum principle for laplace equation is used.

3.13.4 Recovering a Harmonic Function Inside a Ball II

By Using its Normal Derivative on the Boundary

Let $h(y) : W \subseteq \mathbb{R}^n \to R$ be a harmonic function and $W = B(z,r) \subseteq \mathbb{R}^n$ is a ball. Then

$$\phi(y) = \sum_{i=1}^{n} y_i \frac{\partial h(y)}{\partial y_i} \; and \; \frac{\partial h(y)}{\partial y_i}, i \in 1, 2, ..., n \tag{3.334}$$

are harmonic functions on the ball W;

$$\varphi(x) = \varphi \frac{\partial h(x)}{\partial \rho}, \text{ for any x in } \Sigma = \partial W \tag{3.335}$$

where $\frac{\partial h(x)}{\partial \rho}$ is the normal derivative of h on Σ. The property (3.334) is obtained by a direct computation the property (3.335) uses the following argument. Without restricting generality, take $z = 0, r = 1$, and for $y = \rho.x, |x| = 1, 0 \leqslant \rho \leqslant 1$, rewrite h(y) in (3.334) as follows

Theorem 3.13.4. *Let $\lambda(x) : \sum \subseteq \mathbb{R}^n \to \mathbb{R}$ be a continuous function satisfying $\oint\limits_{\Sigma} \lambda(x) dS = 0$, where $\sum = \partial W$ is the boundary of a ball $W = B(z,r) \subseteq \mathbb{R}^n$ .Then there exists a continuous function $h(y) : W \to \mathbb{R}$ which is harmonic on $y \in intW$ and admitting the normal derivative $\frac{\partial h(\rho x)}{\partial \rho}$ $(x \in \sum, 0 < \rho \leqslant 1)$ which equals $\lambda(x)$ for $\rho = 1$. Any other function φ with these properties verifies*

$$\varphi(y) - h(y) = const \; (\forall) \; y \in W. \tag{3.336}$$

Proof. The existence of a function h uses the Poisson kernel $P(x,y)$ given in (§3.10),

$$\varphi(y) = \oint\limits_{\Sigma} P(x,y)\lambda(x) dS. \tag{3.337}$$

Using (§3.16, §3.17) we know that the function $\{\varphi(y), y \in W\}$ is continuous and

$$\begin{cases} \varphi(x) = \lambda(x), & x \in \sum \\ \Delta\varphi(y) = 0, & \text{y in int W (varphi is harmonic on int W)} \end{cases} \tag{3.338}$$

In addition, using $\oint\limits_{\Sigma} \lambda(x) dS = 0$ we get

$$\varphi(0) = \oint\limits_{\Sigma} \lambda(x) dS = 0 \tag{3.339}$$

Define

$$h(y) = h(\rho x) = \int_0^\rho \frac{\varphi(\tau x)}{\tau} d\tau + h(0) \tag{3.340}$$

where $\varphi(y)$ is continuously derivable at $y = 0$, $\varphi(0) = 0$ and $h(0) = const$ (arbitrarily fixed). We shall show that $h(y) : W \to \mathbb{R}$ is a continuous function satisfying

$$\Delta h(y) = 0, \; y \in int\, W \; (h \text{ is harmonic in } int\, W) \tag{3.341}$$

$$\left.\frac{\partial h(\rho x)}{\partial \rho}\right|_{\rho=1} = \frac{\partial h(x)}{\partial \rho} = \varphi(x) = \lambda(x), \; x \in \sum \tag{3.342}$$

For simplicity, take $h(0) = 0$, and using (3.340) for $\rho = 1$ we compute

$$\oint_\Sigma P(s,y)h(s)dS = \oint_\Sigma P(s,y)\Big\{\int_0^1 \frac{\varphi(\tau s)}{\tau} d\tau\Big\} dS \int_0^1 \Big\{\oint_\Sigma P(s,y)\frac{\varphi(\tau s)}{\tau} dS\Big\} d\tau \tag{3.343}$$

By the definition $\frac{\varphi(\tau s)}{\tau}$, $\tau \in [0,1]$ fixed, is a harmonic function satisfying (3.338) and (3.343) is rewritten as

$$\oint_\Sigma P(s,y)h(s)dS = \int_0^1 \frac{\varphi(\tau s)}{\tau} d\tau = \int_0^\rho \frac{\varphi(\xi x)}{\xi} d\xi = h(y). \tag{3.344}$$

where

$$\left\{\frac{\varphi(\tau y)}{\tau}, \tau \in [0,1]\right\} \to \left\{\frac{\varphi(\xi y)}{\xi}, \xi \in [0,\rho]\right\} \tag{3.345}$$

and $\xi = \rho\tau$ are used. The equation (3.344) stands for (3.341) and (3.342) is obtained from (3.340) by a direct derivation. □

Bibliographical Comments

Mainly, it was written using the references [5] and [10]. The last section 3.9 follows the same presentation as in the reference [12]. In the appendices are are used the presentation contained in the references [4] and [9].

Chapter 4

Stochastic Differential Equations; Approximation and Stochastic Rule of Derivations

4.1 Properties of continuous semimartingales and stochastic integrals

Let a complete probability space $\{\Omega, F, P\}$ be given. Assume that a family of sub σ-fields $F_t \subseteq F, t \in [0, T]$, is given such that the following properties are fulfilled:

i) Each F_t contains all null sets of F.

ii) (F_t) is increasing, i.e. $F_t \subseteq F_s$ if $t \geqslant s$.

iii) (F_t) is right continuous, i.e. $\bigcap_{\varepsilon>0} F_{t+\varepsilon} = F_t$ for any $t < T$. Let $X(t), t \in [0, T]$ be a measurable stochastic process with values in R. We will assume, unless otherwise mentioned, that it is F_t-adapted, i.e., $X(t)$ is F_t-measurable for any $t \in [0, T]$. The process $X(t)$ is called continuous if $X(t; \omega)$ is a continuous function of t for almost all $\omega \in \Omega$. Let L_c be the linear space consisting of all continuous stochastic processes. We introduce the metric ρ by $|X - Y| = \rho(X, Y) = (E[\sup_t |X(t) -$

$Y(t)|^2/(1+\sup_t |X(t)-Y(t)|^2])^{1/2}$. It is equivalent to the topology of the uniform convergence in probability. A sequence $\{X^n\}$ of L_c is a Cauchy sequence iff for any $\varepsilon > 0$, $P(\sup_t |X^n(t) - X^m(t)| > \varepsilon) \to 0$ if $n, m \to \infty$. Obviously L_c is a complete metric space. We introduce the norm $||\cdot||$ by $||X|| = (E[\sup_t |X(t)|^2])^{1/2}$ and denote by L_c^2 the set of all elements in L_c with finite norms. We may say that the topology of L_c^2 is the uniform convergence in L^2. Since $\rho(X,0) \leqslant ||X||$, the topology by $||\,||$ is stronger than that by ρ and it is easy to see that L_c^2 is a dense subset of L_c.

Definition 4.1.1. *Let $X(t), t \in [0,T]$, be a continuous F_t-adapted process. (i) It is called a martingale if $E|X(t)| < \infty$ for any t and satisfies $E[X(t)/F_s] = X(s)$ for any $t > s$.*

(ii) It is called a local martingale if there exists an increasing sequence of stopping times (T_n) such that $T_n \uparrow T$ and each stopped process $X_{(t)}^{T_n} \equiv X(t \wedge T_n)$ is a martingale (iii) It is called an increasing process if $X(t;\omega)$ is an increasing function of t almost surely (a.s.) with respect to $\omega \in \Omega$, i.e. there is a P-null set $N \subseteq \Omega$ such that $X(t,\omega)$, $t \in [0,T]$ is an increasing function for any $\omega \in \Omega \setminus N$.

(iv) It is called a process of bounded variation if it is written as the difference of two increasing processes.

(v) It is called a semi-martingale if it is written as the sum of a local martingale and a process of bounded variation.

We will quote two classical results of Doob concerning martingales without giving proofs.(see A. Friedmann)

Theorem 4.1.2. *Let $X(t), t \in [0,T]$ be a martingale.*

(i) **Optional sampling theorem**. *Let S and $\mathcal{U}$ be stopping times with values in $[0,T]$. Then $X(S)$ is integrable and satisfies $E[X(S)/F(\mathcal{U})] = X(S \wedge \mathcal{U})$, where $F(\mathcal{U}) = \{A \in F_T : A \cap \{\mathcal{U} \leqslant t\}\} \in F(t)$ for any $t \in [0,T]\}$*

(ii) **Inequality**. *Suppose $E|X(T)|^p < \infty$ with $p > 1$. Then*

$$E \sup_t |X(t)|^p \leqslant q^p E|X(T)|^p$$

where

$$\frac{1}{p} + \frac{1}{q} = 1$$

Remark 4.1.3. *Let S be a stopping time. If $X(t)$ is a martingale the stopped process $X^S(t) \equiv X(t \wedge S)$ is also a martingale. In fact, by Doob's optional sampling theorem*

we have (for $t \geqslant s$) $E[X^S(t)|F(s)] = X(t \wedge S \wedge s) = X(S \wedge s) = X^S(s)$. Similarly, if X is a local martingale the stopped process X^S is a local martingale. Let X be a local martingale. Then there is an increasing sequence of stopping times $S_k \uparrow T$ such that each stopped process X^{S_k} is a bounded martingale. In fact, define S_k by $S_k = \inf\{t > 0 : |X(t)| \geqslant k\}(= T$ if $\{\cdots\} = \phi)$. Then $S_k \uparrow T$ and it holds $\sup\limits_t |X^{S_k}(t)| \leqslant k$, so that each X^{S_k} is a bounded martingale.

Remark 4.1.4. *Let M_c be the set of all square integrable martingale, $X(t)$ with $X(0) = 0$. Because of Doob's inequality the norm $||X||$ is finite for any $X \in M_c$. Hence M_c is a subset of L_c^2. We denote M_c^{loc} the set of all continuous local martingales $X(t)$ such that $X(0) = 0$; it is a subset of L_c.*

Theorem 4.1.5. *M_c is a closed subspace of $L_c^2 \cdot M_c^{\text{loc}}$ is a closed subspace of L_c. Further more, M_c is dense in M_c^{loc}.*

Remark 4.1.6. *Denote by H_t^2 the set consisting of all random variables $h : \Omega \to R$ which are F_t-measurable and $E|h|^2 < \infty$. Then for each $X \in L_c^2$ there holds $E[X(T)/F_t] = \widehat{h}$ where $\widehat{h} \in H_t^2$ is the optional solution for*

$$\min_{h \in H_t^2} E|X(T) - h|^2 = E|X(T) - \widehat{h}|^2$$

Definition *Let $X(t)$ be a continuous stochastic process and Δ a partition of the interval $[0,T] : \Delta = \{0 = t_0 < \ldots < t_n = T$, and let $|\Delta| = max(t_{i+1} - t_i)$. Associated with the partition Δ, we define a continuous process $\langle X\rangle^\Delta(t)$ as*

$$\langle X\rangle^\Delta(t) = \sum_{i=0}^{k-1}(X(t_{i+1} - X(t_i))^2 + (X(t) - X(t_k))^2$$

where k is the number such that $t_k \leqslant t < t_{k+1}$. We call it the quadratic variation of $X(t)$ associated with the partition Δ Now let $\{\Delta\}_m$ be a sequence of partitions such that $|\Delta_m| \to 0$. If the limit of $\langle X\rangle^{\Delta_m}(t)$ exists in probability and it is independent of the choice of sequence $\{\Delta_m\}$ a.s., it is called the quadratic variation of $X(t)$ and is denoted by $\langle X\rangle(t)$. We will see that a natural class of processes where quadratic variations are well defined is that of continuous semimartingales.

Lemma 4.1.7. *Let X be a continuous process of bounded variation. Then the quadratic variation exists and it equals zero a.s.*

Proof. Let $|X|(t;\omega)$ be the total variation of the function $X(s,\omega), 0 \leqslant s \leqslant t$. Then there holds

$$|X|(t) = \sup_\Delta[|X(t_0)| + \sum_{j=0}^{k-1}|X(t_{j+1}) - X(t_j)| + |X(t) - X(t_k)|]$$

and $\langle X\rangle^\Delta(t) \leqslant \left(\sum_{j=0}^{k-1}|X(t_{j+1}) - X(t_j)| + |X(t) - X(t_k)|\right)\max\limits_i |X(t_{i+1}) - X(t_i)|$
$\leqslant |X|(t)\max\limits_i |X(t_{i+1}) - X(t_i)|$

The right hand side converges to 0 for $|\Delta| \to 0$ a.s. $\square$

Theorem 4.1.8. *Let $M(t)\, t \in [0,T]$, be a bounded continuous martingale. Let $\{\Delta_n\}$ be a sequence of partitions such that $|\Delta_n| \to 0$. Then $\langle M\rangle^{\Delta_n}(t)$, $t \in [0,T]$, converges uniformly to a continuous increasing process $\langle M\rangle(t)$ in L^2 sense, i.e.,*

$$\lim_{n\to\infty} E[\sup_t |\langle M\rangle^{\Delta_n}(t) - \langle M\rangle(t)|^2] = 0;$$

in addition $M^2(t) - \langle M\rangle(t), t \in [0,T]$ is a martingale.

The proof is based on the following two lemmas.

Lemma 4.1.9. *For any $t > s$, there holds,*

$$E[\langle M\rangle^{\Delta}(t)/F(s)] - \langle M\rangle^{\Delta}(s) = E[(M(t) - M(s))^2|F(s)]$$

In particular, $M^2(t) - \langle M\rangle^{\Delta}(t)$ is a continuous martingale.

Hint. Rewrite $M(t)$ and $M(s)$ as follows $(s < t)$

$$M(t) = M(t) - M(t_k) + \sum_{i=0}^{k-1}[M(t_{i+1} - M(t_i)] + M(0), \text{where}\, t_k \leqslant t < t_{k+1}$$

$$M(s) = M(s) - M(s_l) + \sum_{i=0}^{l-1}[M(s_{i+1}) - M(s_i)] + M(0), \text{where}\, s_l \leqslant s < s_{l+1}$$

and $\Delta \cap [0,s] = \{0 = s_0 < s_1 < \ldots < s_l\}$. On the other hand

$$\begin{aligned} E[(M(t) - M(s))^2/F(s)] &= E[(M(t) - M(t_k))^2 + \sum_{i=l+1}^{k-1}(M(t_{i+1}) - M(t_i))^2 \\ &+ (M(s_{l+1}) - M(s))^2/F(s)] \end{aligned} \quad (4.1)$$

where $t_{l+1} = s_{l+1}$ and

$$\begin{aligned} &E[(M(t_{i+1}) - M(t_i))(M(t_{j+1}) - M(t_j)]/F(s)] \\ =\; &E\{E[(M(t_{i+1}) - M(t_i)](M(t_{j+1}) - M(t_j))/F(t_{j+1})]/F(s)\} = 0\,(\text{if} i > j) \end{aligned} \quad (4.2)$$

are used. Adding and subtracting $M^{\Delta}(s) = M(s) - M(s_l))^2 + \sum_{i=0}^{l-1}(M(s_{i+1}) - M(s_i))^2$ we get the first conclusion which can be written as a martingale property.

Lemma 4.1.10. *It holds*

$$\lim_{n,m\to\infty} E[|\langle M\rangle^{\Delta_n}(T) - \langle M\rangle^{\Delta_m}(T)|^2] = 0.$$

Theorem 4.1.11. *$M(t)$ be a continuous local martingale. Then there is a continuous increasing process $\langle M\rangle(t)$ such that $\langle M\rangle^{\Delta}(t)$ converges uniformly to $\langle M\rangle(t)$ in probability.*

Corollary 4.1.12. *$M^2(t) - \langle M\rangle(t)$ is a local martingale if $M(t)$ is a continuous local martingale.*

Theorem 4.1.13. *Let $X(t)$ be a continuous semi-martingale. Then $\langle X\rangle^{\Delta}(t)$ converges, uniformly to $\langle M\rangle(t)$ in probability as $|\Delta| \to 0$, where $M(t)$ is the local martingale part of $X(t)$*

Stochastic integrals Let $M(t)$ be a continuous local martingale and let $f(t)$ be a continuous $F(t)$ adapted process. We will define the stochastic integral of $f(t)$ by the differential $dM(t)$ using the properties of martingales, especially those of quadratic variations. Let $\Delta = (0 = t_0 < \ldots < t_n = T)$ be a partition of $[0, T]$. For any $t \in [0, T]$ choose t_k of Δ such that $t_k \leqslant t < t_{k+1}$ and define

$$L^{\Delta}(t) = \sum_{i=0}^{k-1} f(t_i)(M(t_{i+1}) - M(t_i)) + f(t_k))(M(t) - M(t_k)) \tag{4.3}$$

It is easily seen that $L^{\Delta}(t)$ is a continuous local martingale. The quadratic variation is computed directly as

$$\begin{aligned} \langle L^{\Delta}\rangle(t) &= \sum_{i=0}^{k-1} f^2(t_i)(\langle M\rangle(t_{i+1}) \\ &- \langle M\rangle(t_i)) + f^2(t_k)(\langle M\rangle(t) - \langle M\rangle(t_k)) \int_0^t |f^{\Delta}(s)|^2 d\langle M\rangle(s) \end{aligned} \tag{4.4}$$

where $f^{\Delta}(s)$ is a step process defined from $f(s)$ by $f^{\Delta}(s) = f(t_k)$ if $t_k \leqslant s < t_{k+1}$. Let Δ' be another partition of $[0, T]$. We define $L^{\Delta'}(t)$ similarly, using the same $f(s)$ and $M(s)$. Then there holds

$$\langle L^{\Delta} - Ł^{\Delta'}\rangle(t) = \int_0^t |f^{\Delta}_{(s)} - f^{\Delta'}_{(s)}|^2 d\langle M\rangle(s).$$

Now let $\{\Delta_n\}$ be a sequence of partitions of $[0, T]$ such that $|\Delta_n| \to 0$. Then $\langle L^{\Delta_n} - L^{\Delta_m}\rangle(T)$ converges to 0 in probability as $n, m \to \infty$. Hence $\{L^{\Delta_n}\}$ is a Cauchy sequence in M_c^{loc}. We denote the limit as $L(t)$.

Definition 4.1.14. *The above $L(t)$ is called the Itô integral of $f(t)$ by $dM(t)$ and is denoted by $L(t) = \int_0^t f(s) dM(s)$.*

Definition 4.1.15. *Let $X(t)$ be a continuous semimartingale decomposed to the sum of a continuous local martingale $M(t)$ and a continuous process of bounded variation $A(t)$*

Let f be an $F(t)$-adapted process such that $f \in L^2(\langle M\rangle)$ and $\int_0^T |f(s)|d|A|(s)\langle\infty$. Then the Ito integral of $f(t)$ by $dX(t)$ is defined as

$$\int_0^t f(s)dX(s) = \int_0^t f(s)dM(s) + \int_0^t f(s)dA(s)$$

Remark 4.1.16. *If f is a continuous semimartingale then $\int_0^t f(s)dX(s)$ exists. We will define another stochastic integral by the differential "$\circ dX$ (s) $s \in [0,t]$ $\int_0^t f(s) \circ dX(s) = \lim_{|\Delta|\to 0}\Big[\sum_{i=0}^{k-1} \frac{1}{2}(f(t_{i+1}) + f(t_i))(X(t_{i+1}) - X(t_i))$
$+ \frac{1}{2}(f(t) + f(t_k))(X(t) - X(t_k))\Big]$*

Definition 4.1.17. *If the above limit exists, it is called the Fisk-Stratonovich integral of $f(s)$ by $dX(s)$.*

Remark 4.1.18. *If f is a continuous semimartingale, the Fisk-Stratonovitch integral is well defined and satisfies $\int_0^t f(s) \circ dX(s) = \int_0^t f(s)dX(s) + \frac{1}{2}\langle f, X\rangle(t)$*

Proof. Is easily seen from the relation

$$\begin{aligned} & \sum_{i=0}^{k-1} \frac{1}{2}(f(t_{i+1}) + f(t_i))(X(t_{i+1}) - X(t_i)) + \frac{1}{2}(f(t) + f(t_k))(X(t) - X(t_k)) \\ = & \sum_{i=0}^{k-1} f(t_i)(X(t_{i+1}) - X(t_i)) + f(t_k)(X(t) - X(t_k)) + \frac{1}{2}\langle f, X\rangle^{\Delta}(t) \end{aligned} \tag{4.5}$$

where the joint quadratic variation

$$\langle f, X\rangle^{\Delta}(t) = \sum_{i=0}^{k-1}(f(t_{i+1} - f(t_i))(X(t_{i+1} - X(t_i)) + (f(t) - f(t_k))(X(t) - X(t_k))$$

and k is the number such that $t_k \leqslant t < t_{k+1}$. ☐

Theorem 4.1.19. *Let X and Y be continuous semi-martingales. The joint quadratic variation associated with the partition Δ is defined as before and is written $\langle X, Y\rangle^{\Delta}$. Then $\langle X, Y\rangle^{\Delta}$ converges uniformly in probability to a continuous process of bounded*

variation $\langle X, Y\rangle(t)$. If M and N are local martingale parts of X and Y, respectively, then $\langle X, Y\rangle$ coincides with $\langle M, N\rangle$

Remark 4.1.20. *Let $w(t) = (w^1(t), \ldots, w^m(t))$ be an m-dimensional $(F_{(t)})$-adapted continuous stochastic process. It is an $(F_{(t)})$-Wiener process iff each $w^i(t)$ is a scalar Wiener process with the joint quadratic variation fulfilling $\langle w_i, w_j\rangle(t) = \delta_{ij}t$ where $\delta ii = 1$ and $\delta_y = 0$ $i \neq j$*

4.2 Approximations of the diffusion equations by smooth ordinary differential equations

The m-dimensional Wiener process is approximated by a smooth process and it allows one to use non anticipative smooth solutions of ordinary differential equations as approximations for solutions of a diffusion equation. It comes from Langevin's classical procedure of defining a stochastic differential equation. By a standard m-dimensional Wiener process we mean a measurable function $w(t, \omega) \in R^m$, $(t; \omega) \in [0, \infty) \times \Omega$ with continuous trajectories $w(t, \omega)$ for each $\omega \in \Omega$ such that $w(0, \omega) = 0$, and i_1) $E(w(_2)) - w(t_1)/F_{t_1}) = 0$, i_2) $E([w(t_2) - w(t_1)][w(t_2) - w(t_1)]^T/\mathcal{F}_{t_1}) = I_m(t_2 - t_1)$ for any $0 \leqslant t_1 < t_2$, where $(\Omega, \mathcal{F}, P)$ is a given complete probability space and $F_t \subseteq F$ is the σ-algebra generated by $(w(s), s \leqslant t)$. The quoted Langevin procedure replaces a standard m-dimensional Wiener process by a C^1 non-anticipative process $v_\varepsilon(t)$ (see $v_\varepsilon(t)$ is F_t measurable) as follows

$$v_\varepsilon(t) = w(t) - \int_0^t (\exp - \beta(t-s))dw(s),\ \beta = \frac{1}{\varepsilon}, \varepsilon \downarrow 0 \tag{4.6}$$

where the integral in the right hand side is computed as

$$w(t) - \beta \int_0^t w(s)(\exp - \beta(t-s))ds\ \text{ (the integration by parts formula)} \tag{4.7}$$

Actually, $v_\varepsilon(t)$, $t \in [0, T]$, is the solution of the following equations

$$\frac{dv_\varepsilon(t)}{dt} = -\beta v_\varepsilon(t) + \beta w(t) = \beta(w(t) - v_\varepsilon(t),\ v_\varepsilon(0) = 0 \tag{4.8}$$

and by a direct computation we obtain

$$E||v_\varepsilon(t) - w(t)||^2 \leqslant \varepsilon,\ t \in [0, T] \tag{4.9}$$

Rewrite $v_\varepsilon(t)$ in (4.6) as

$$v_\varepsilon(t) = w(t) - \eta_\varepsilon(t),$$

where

$$\eta_\varepsilon(t) = \int_0^t \exp - \beta(t-s)dw(s) \tag{4.10}$$

fulfills $d\eta^i = -\beta\eta^i dt + dw^i(t),\ i = 1, \dots, m$, and

$$\frac{dv_\varepsilon}{dt}(t) = \beta\eta_\varepsilon(t),\ \beta = \frac{1}{\varepsilon},\ \varepsilon \downarrow 0. \tag{4.11}$$

Now, we are given continuous functions

$$f(t,x), g_j(t,x) : [0,T] \times R^n \to R^n,\ j = 1, \dots, m,$$

such that

$$(\alpha) \begin{cases} i)\ f, g_j \text{ are bounded}, \quad j = 1, \dots, m \\ ii)\ ||h(t,x'') - h(t,x')|| \leqslant L||x'' - x'||\ (\forall)x', x'' \in R^n, t \in [0,T] \end{cases}$$

where $L > 0$ is a constant,$h \triangleq f, g_j$. In addition, we assume that ($g_j \in C_b^{1,2}([0,T] \times R^n)$
β) $\frac{d\partial g_j}{\partial t}, \frac{d\partial g_j}{\partial x}, \frac{d\partial^2 g_j}{\partial t \partial x}, \frac{d\partial^2 g_j}{\partial x^2}, j = 1, \dots, m$, are continuous and bounded functions. Let $x_0(t)$ and $x_\varepsilon(t), t \in [0,T]$, be the solution in

$$dx = [f(t,x) + \frac{1}{2}\sum_{i=1}^m \frac{\partial g_i}{\partial x}(t,x)g_i(t,x)]dt \tag{4.12}$$

$$+ \sum_{i=1}^m g_i(t,x)dw^i_{(t)},\ x(0) = x_0$$

and

$$\frac{dx}{dt} = f(t,x) + \sum_{i=1}^m g_i(t,x)\frac{dv^i_\varepsilon(t)}{dt},\ x(0) = x_0 \tag{4.13}$$

correspondingly, where $v_\varepsilon(t)$ is associated with $w(t)$ in (4.6) and fulfills (4.8)-(4.11). It is the Fisk-Stratonovich integral (see "$\circ$" below) which allow one to rewrite the system (4.12) as

$$dx = f(t,x)dt + \sum_{i=1}^m g_i(t,x) \circ dw^i(t), x(0) = x_0, \tag{4.14}$$

where

$$g_i(t,x) \circ dw^i(t) \stackrel{def}{=} g_i(t,x)dw^i(t) + \frac{1}{2}\frac{\partial g_i}{\partial x}(t,x)g_i(tx)dt$$

Theorem 4.2.1. *Assume that continuous functions $f(t,x), g_i(t,x), t \in [0,T], x \in R^n$, are given such that (α) and (β) are fulfilled. Then $\lim_{\varepsilon \to 0} E||x_\varepsilon(t) - x_0(t)||^2 = 0$, $t \in [0,T]$, where $x_0(t)$ and $x_\varepsilon(t)$ are the solutions defined in* (4.12) *and, respectively,*(4.13)

Proof. Using (4.12) we rewrite the solution $x_\varepsilon(t)$ as

$$x_\varepsilon(t) = x_0 + \int_0^t f(s), x_\varepsilon(s))ds + \sum_{i=1}^m \beta \int_0^t \eta_\varepsilon^i(s) g_i(s, x_\varepsilon(s))ds, \ t \in [0,T] \tag{4.15}$$

and from F_t-measurability of $\eta_\varepsilon(t)$ we obtain that $x_\varepsilon(t)$ is $\mathcal{F}_t$-measurable and non-anticipative with respect to $\{\mathcal{F}_t\}$, $\quad t \in [0,T]$. Therefore, the stochastic integrals $\int_0^t g_i(s, x_\varepsilon(s))dw^i(s)$ and $\int_0^t g_i(s, x_\varepsilon(s))d\eta_\varepsilon^i(s)$ are well defined, and using (4.10) we obtain

$$\int_0^t g_i(s, x_\varepsilon(s))d\eta_\varepsilon^i(s) = -\beta \int_0^t g_i(s, x_\varepsilon(s))\eta_\varepsilon^i(s)ds + \int_0^t g_i(s, x_\varepsilon(s, x_\varepsilon(s))dw^i(s). \tag{4.16}$$

Step 1

Using (4.15)in(4.15) there follows

$$\begin{aligned} x_\varepsilon(t) &= x_0 + \int_0^t f(s, x_\varepsilon(s))ds + \sum_{i=1}^m \int_0^t g_i(s, x_\varepsilon(s))dw^i(s) \\ &- \sum_{i=1}^m \int_0^t g_i(s_s, x_\varepsilon(s))d\eta_\varepsilon^i(s), \ t \in [0,T] \end{aligned} \tag{4.17}$$

In what follows it will be proved that (see step 2 and step 3)

$$-\int_0^t g_i(s, x_\varepsilon(s))d\eta_\varepsilon^i(s) = \frac{1}{2}\int_0^t \frac{\partial g_i}{\partial x}(s, x_\varepsilon(s))g_i(s, x_\varepsilon(s))ds + O_t(\varepsilon) \tag{4.18}$$

where $E||O_t(\varepsilon)||^2 \leqslant c_1\varepsilon, \ (\forall)\, t \in [0,T]$, for some constant $c_1 > 0$. Using (4.18) in (4.17) we rewrite (4.17) as

$$\begin{aligned} x_\varepsilon(t) &= x_0 + \int_0^t \Big[f(s, x_\varepsilon(s)) \\ &+ \frac{1}{2}\sum_{i=1}^m \frac{\partial g_i}{\partial x}(s, x_\varepsilon(s))g_i(s, x_\varepsilon(s))\Big] ds \sum_{i=1}^m \int_0^t g_i(s, x_\varepsilon(s))dw^i(s) + O_t(\varepsilon), \end{aligned} \tag{4.19}$$

where $E|O_t(\varepsilon)||^2 \leqslant c_1\varepsilon$.

The hypotheses (α) and (β) allow one to check that

$$\widetilde{f}(t,x) \triangleq f(t,x) + \frac{1}{2}\sum_{i=1}^m \frac{\partial g_i}{\partial x}(t,x)g_i(t,x)$$

and $g_i(t,x)$ fulfil (α) also but with a new Lipschitz constant $\widetilde{L}$.

The proof will be complete noticing that

$$E||x_\varepsilon(t) - x_0(t)||^2 \leqslant c_2 \int_0^t E||x_\varepsilon(s) - x_0(s)||^2 ds + c_3\varepsilon,\ t \in [0,T] \tag{4.20}$$

for some constants $c_2, c_3 > 0$ and Gronwall's lemma applied to (4.20) implies

$$E||x_\varepsilon(t) - x_0(t)||^2 \leqslant \varepsilon c_3(\exp T_{c_2}) = \varepsilon c_4 \tag{4.21}$$

which proves the conclusion.

Step 2

To obtain (4.18) fulfilled we use an ordinary calculus as integration by parts formula (see $x_\varepsilon(t)$ is a C^1 function in $t \in [0,T]$). More precisely

$$\begin{aligned} -\int_0^t g_i(s, x_\varepsilon(s)) d\eta_\varepsilon^i(s) &= -g_i(t, x_\varepsilon(t)\eta_\varepsilon^i(t) + \int_0^t \eta_\varepsilon^i(s)\Big(\frac{d}{ds} g_i(s, x_\varepsilon(s))\Big) ds \\ &= \sum_{j=1}^m \int_0^t \frac{\partial g_i}{\partial x}(s, x_\varepsilon(s)) g_j(s, x_\varepsilon(s)) \frac{dv_\varepsilon^j(s)}{ds} \eta_\varepsilon^i(s) ds \\ &+ \quad O_t^i(\varepsilon) \triangleq \sum_{j=1}^m T_{ij}(t) + O_t^i(\varepsilon), \end{aligned} \tag{4.22}$$

Here

$$T_{ij}(t) \triangleq \int_0^t \frac{\partial g_i}{\partial x}(s, x_\varepsilon(s)) g_j(s, x_\varepsilon(s)) \frac{dv_\varepsilon^j(s)}{ds} \eta_\varepsilon^i(s) ds \ \text{ and}$$

$$O_t^i(\varepsilon) \triangleq \int_0^t \eta_\varepsilon^i(s)\Big[\frac{\partial g_i}{\partial s}(s, x_\varepsilon(s)) + \frac{\partial g_i}{\partial x}(s, x_\varepsilon(s)) f(s, x_\varepsilon(s))\Big] ds \\ -g_i(t, x_\varepsilon(t))\eta_\varepsilon^i(t)$$

satisfies

$$E||O_t^1(\varepsilon)||^2 \leqslant k\varepsilon,\ t \in [0,T], \tag{4.23}$$

(see $E||\eta_\varepsilon^i(t)||^2 \leqslant \varepsilon$ in (4.10) and $f, g_i, \dfrac{\partial g_i}{\partial x}; \dfrac{\partial g_i}{\partial t}$ are bounded). On the other hand, $\dfrac{dv_\varepsilon^j}{dt} = \beta\eta_\varepsilon^j(t)$ (see (4.11)) and using $\beta\eta_\varepsilon^j(t)dt = dw^j(t) - d\eta_\varepsilon^j(t)$ (see (4.10)) we obtain that T_{ij} in (4.22) fulfils

$$T_{ij}(t) = -\int_0^t g_{ij}(s, x_\varepsilon(s))\eta_\varepsilon^i(s) d\eta_\varepsilon^j(s) + \theta_t^2(\varepsilon), \tag{4.24}$$

where $g_{ij}(t,x) \triangleq \dfrac{\partial g_i}{\partial x}(t,x)g_j(t,x)$ and

$$\theta_t^2(\varepsilon) \triangleq \int_0^t g_{ij}(s,x_\varepsilon(s))\eta_\varepsilon^i(s)dw^j(s)$$

satisfies

$$E||\theta_t^2(\varepsilon)||^2 \leqslant k_2\varepsilon,\ t\in[0,T] \tag{4.25}$$

(see g_{ij} bounded and $E||\eta_\varepsilon^i(t)||^2 \leqslant \varepsilon$).

<u>*Step 3*</u>

(a) For $i=j$ we use formulas

$$\eta_\varepsilon^i(t) = (\exp-\beta t)\mu_\varepsilon^i(t), \mu_\varepsilon^i(t) = \int_0^t (\exp\beta s)dw^i(s)$$

$$\begin{aligned}(\eta_\varepsilon^i(t))^2 = \quad & (\exp-2\beta t)(\mu_\varepsilon^i(t))^2 = (\exp-2\beta t)\Big[\int_0^t 2(\exp2\beta s)\eta_\varepsilon^i(s)dw^i(s) + \int_0^t(\exp2\beta s)ds\Big]\\ = \quad & 2\int_0^t \eta_\varepsilon^i(s)d\eta_\varepsilon^i(s) + \int_0^t ds\end{aligned}$$

$$T_{ii}(t) = -\int_0^t g_{ii}(s,x_\varepsilon(s))\eta_\varepsilon^i(s)d\eta_\varepsilon^i(s) + \theta_t^2(\varepsilon) \text{ (see (4.24))}$$

We get

$$\begin{aligned}T_{ii}(t) &= \frac{1}{2}\int_0^t g_{ii}(s,x_\varepsilon(s))ds \ -\frac{1}{2}\int_0^t g_{ii}(s,x_\varepsilon)d(\eta_\varepsilon^i(t))^2 + \theta_t^2(\varepsilon)\\ &= \frac{1}{2}\int_0^t g_{ii}(s,x_\varepsilon(s))ds \ -\frac{1}{2}g_{ii}(t,x_\varepsilon(t))(\eta_\varepsilon^i(t))^2\\ &+ \frac{1}{2}\int_0^t \Big[\frac{d}{ds}g_{ii}(s,x_\varepsilon(s))\Big](\eta_\varepsilon^i(s))^2ds + \theta_t^2(\varepsilon)\\ &= \frac{1}{2}\int_0^t g_{ii}(s,x_\varepsilon(s))ds + O_t^2(\varepsilon).\end{aligned} \tag{4.26}$$

Here

$$\begin{aligned}O_t^2(\varepsilon) \triangleq\ & \theta_t^2(\varepsilon) - \frac{1}{2}g_{ii}(t,x_\varepsilon(t))(\eta_\varepsilon^i(t))^2\\ +\frac{1}{2}\int_0^t & \Big[\frac{\partial}{\partial s}g_{ii}(s,x_\varepsilon(s)) + \frac{\partial g_{ii}}{\partial x}(s,x_\varepsilon(s))f(s,x_\varepsilon(s))\\ +\beta & \sum_{j=1}^m \frac{\partial g_{ii}}{\partial x}(s,x_\varepsilon(s))g_j(s,x_\varepsilon(s))\eta_\varepsilon^j(s)\Big](\eta_\varepsilon^i(s))^2ds\end{aligned}$$

fulfils $E||O_t^2(\varepsilon)||^2 \leqslant \widetilde{K}_2\varepsilon$ taking into account that

$$E||\theta_t^2(\varepsilon)||^2 \leqslant k_2\varepsilon \,(\text{see } (4.24)),\;\; E(\eta_\varepsilon^i(t))^2 = (\exp -2\beta t)E(\mu_\varepsilon^i(t))^2 \leqslant \varepsilon \qquad (4.27)$$

In addition, write $(\eta_\varepsilon^i(t))^2 = (\exp -2\beta t)(\mu_\varepsilon^i(t))^2$, where $(\mu_\varepsilon^i(t))^2 = a_i(t) + b(t)$ is the corresponding decomposition using martingale $a_i(t) \triangleq \int_0^t 2\mu^i(s)(\exp\beta s)dw^i(s)$ and $b(t) \triangleq \int_0^t (\exp 2\beta s)ds$. Compute $(\eta_\varepsilon^i(t))^4 = (\exp -4\beta t)[a_i(t))^2 + (b(t))^2 + 2a_i(t)b(t)]$ and $(\eta_\varepsilon^i(t))^6 = (\exp -6\beta t)[(a_i(t))^3 + (b(t))^3 + 3(a_i(t))^2 b(t) + 3a_i(t)(b(t))^2]$. Noticing that $a_i(t)$, $t \in [0,T]$, is a martingale with $a_i(0) = 0$ we can prove (exercise!) that any $(a_i(t))^m (m = 2,3)$ satisfies

$$E(a_i(t))^m = \frac{1}{2}\int_0^t m\,(m-1)\,E[a_i(s))^{m-2}\,(f_i(s))^2]ds$$

where $f_i(t) \triangleq 2(\exp\beta t)\mu^i(t)$ and $a_i(t) \triangleq \int_0^t f_i(s)dw^i(s)$. We get

$$E(\eta_\varepsilon^i(t))^4 \leqslant \,(\text{const})\,\varepsilon^2,\; E(\eta_\varepsilon^i(t))^6 \leqslant (\text{const})\,\varepsilon^3\, t \in [0,T] \qquad (4.28)$$

and the conclusion $E||\theta_t^2(\varepsilon)||^2 \leqslant$ (const)ε used in (4.26) follows directly from (4.27) and (4.28).

(b) For any $i \neq j$, there holds

$$T_{ij}(t) = -\int_0^t g_{ij}(s, x_\varepsilon(s))\eta_\varepsilon^i(s)d\eta_\varepsilon^j(s) + \theta_t^2(\varepsilon)\,(\text{see } (17))$$

where

$$d\eta_\varepsilon^j(s) = -\beta\eta_\varepsilon^j(s)ds + dw^j(s)\,(\text{see } (4.10)) \qquad (4.29)$$

Using (4.29) in (4.24) for $i \neq j$ we obtain that $\eta_\varepsilon^i(t), \eta_\varepsilon^j(t)$ are independent random variables and

$$T_{ij}(t) = -\frac{1}{2}\int_0^t g_{ij}(s, x_\varepsilon(s))d[\eta_\varepsilon^i(s)\eta_\varepsilon^j(s)] + \theta_t(\varepsilon) + \widetilde{O}_t(\varepsilon), \qquad (4.30)$$

where

$$\widetilde{O}_t(\varepsilon) = -\int_0^t g_{ij}(s, x_\varepsilon(s))\eta_\varepsilon^i(s)dw^j(s)$$

satisfies

$$E||\widetilde{O}_t(\varepsilon)||^2 \leqslant (\text{const})\varepsilon\,(\text{see}\, g_{ij} \text{ is bounded and}\, E(\eta_\varepsilon^i(s))^2 \leqslant \varepsilon) \qquad (4.31)$$

Integrating by parts the first term in (4.30) and using

$$E(\eta_\varepsilon^i(t))^2(\eta_\varepsilon^j(t))^2 \leqslant (\text{const})\varepsilon^2,\;\; E(\eta_\varepsilon^i(t))^2(\eta_\varepsilon^j(t))^2(\eta_\varepsilon^k(t))^2 \leqslant (\text{const})\varepsilon^3 \qquad (4.32)$$

we finally obtain

$$T_{ij}(t) = \widetilde{O}_t^2(\varepsilon)\text{with}\, E||\widetilde{O}_t^2(\varepsilon)||^2 \leqslant \widetilde{C}\varepsilon,\; t \in [0,T] \qquad (4.33)$$

Using (4.26), (4.33) in (4.22) we find (4.18) fulfilled. The proof is complete □

Remark 4.2.2. *Under the conditions in theorem* (4.2.1) *it might be useful to notice that the computations remain unchanged if a stopping time* $\tau : \Omega \to [0, T]$, $\{\omega : \tau \geqslant t\} \in \mathcal{F}_t$, $(\forall) t \in [0, T]$, *is used. Namely*

$$\lim_{\varepsilon \downarrow 0} E||x_\varepsilon(t \wedge \tau) - x_0(t \wedge \tau)||^2 = 0 \, (\forall) \, t \in [0, T],$$

if the random varieble $\tau : \Omega \to [0, T]$ *is adapted to* $\{\mathcal{F}_t\}$ *i.e.* $\{\omega : \tau \geqslant t\} \in \mathcal{F}_t$ *for* $t \in [0, T]$.

Proof. By definition

$$x_\varepsilon(t \wedge \tau) = x_0 + \int_0^{t \wedge \tau} f(s, x_\varepsilon(s))ds + \sum_{j=1}^m \int_0^{t \wedge \tau} g_j(s, x_\varepsilon(s)\frac{dv_\varepsilon^j}{ds}(s),$$

$$x_0(t \wedge \tau) = x_0 + \int_0^{t \wedge \tau} \widetilde{f}(t, x_0(s))ds + \sum_{j=1}^m \int_0^{t \wedge \tau} g_j(s, x_0(s)dw^j(s), \, t \in [0, T],$$

where

$$\widetilde{f}(t, x) \triangleq f(t, x) + \frac{1}{2} \sum_{j=1}^m \frac{\partial g_j}{\partial x}(t, x) g_j(t, x).$$

Using the characteristic function

$$\chi(t, \omega) = \begin{cases} 1 \text{ if } \tau(\omega) \geqslant t \\ 0 \text{ if } \tau(\omega) < t \end{cases}$$

which is a non-anticipative function, we rewrite $y_\varepsilon(t) \triangleq x_\varepsilon(t \wedge \tau)$ and $y_0(t) \triangleq \chi_0(t \wedge \tau)$ as

$$(*) \qquad y_\varepsilon(t) = x_0 + \int_0^t \chi(s) f(s, y_\varepsilon(s)ds + \sum_{j=1}^m \int_0^t \chi(s) g_j(s, y_\varepsilon(s)\frac{dv_\varepsilon^j(s)}{ds},$$

$$(**) \qquad y_0(t) = x_0 + \int_0^t \chi(s)\widetilde{f}(s, y_0(s))ds + \sum_{j=1}^m \int_0^t \chi(s) g_j(s, y_0(s))dw^j(s).$$

Now the computations in Theorem 4.2.1 repeated for (*) and (**) allow one to obtain the conclusion. □

Remark 4.2.3. *Using Remark 1 we may remove the boundedness assumption of* f, g_j *in the hypothesis* (α) *of Theorem 4.2.1. That is to say, the solutions* $x_\varepsilon(t), x_0(t)$, $t \in$

$[0,T]$, exist assuming only the hypothesis $(\alpha,(ii))$ and (β), and to obtain the conclusion we multiply f, g_j by a C^∞ scalar function $0 \leqslant \alpha_N(x) \leqslant 1$ such that $\alpha_N(x) = 1$ if $x \in S_N(x_0)$, $\alpha_N(x) = 0$ if $x \in R^n$ $S_{2N}(x_0)$ where $S_\rho(x_0) \subseteq R^n$ is the ball of radius ρ and centered at x_0. We obtain new bounded functions

$$f^N(t,x) = f(t,x)\alpha^N(x),\ g_j^N(t,x) = g_j(t,x)\alpha^N(x)$$

fulfilling (α) and (β) of Theorem 4.2.1, and therefore $\lim_{\varepsilon\downarrow 0} E||x_\varepsilon^N(t) - x_0^N(t)||^2 = 0$ $(\forall)\, t \in [0,T]$, where $x_\varepsilon^N(t)$ and $x_0^N(t)$ are the corresponding solutions. On the other hand, using a stopping time (exit ball time)

$$\tau_N(\omega) = \inf\{t \geqslant 0 \,:\, x_0(t,\omega) \notin S_N(x_0)\}$$

we obtain $x_0(t\wedge\tau_N) = x_0^N(t\wedge\tau_N)$, $t \in [0,T]$ (see Friedmann) where $x_0(t), t \in [0,T]$ is the solution of the equation (4.12) with f, g_i fulfilling (α, ii) and (β).

Finally we obtain:

c) $$\lim_{\varepsilon\to 0} E||x_\varepsilon^N(t\wedge\tau_N) - x_0(t\wedge\tau_N)||^2 = 0$$

for any $t \in [0,T]$, and for arbitrarily fixed $N > 0$. The conclusion (c) represents the approximation of the solution in (4.12) under the hypotheses (α, ii) and (β).

Remark 4.2.4. *The nonanticipative process $v_\varepsilon(t)$, $t \in [0,T]$, used in Theorem 4.2.1 is only of the class C^1 with respect to $t \in [0,T]$, but a minor change in the approximating equations as follows ($\beta = \frac{1}{\varepsilon}$, $\varepsilon \downarrow 0$)*

$$\frac{dv_\varepsilon}{dt}(t) = y_1,\ \varepsilon\frac{dy_1}{dt} = -y_1 + y_2, \ldots,\ \varepsilon\frac{dy_{k-1}}{dt} = -y_{k-1} + y_k$$

$$\varepsilon dy_k = -y_k dt + dw(t),\ t \in [0,T],\ y_j(0) = 0,\ j = 1,\ldots k, v_\varepsilon(0) = 0$$

will allow one to obtain a non-anticipative $v_\varepsilon(t), t \in [0,T]$ of the class C^k, for an arbitrarily fixed k.

4.3 Stochastic rule of derivation

The results contained in Theorem 4.2.1 and in remarks following the theorem give the possibility to obtain, in a straight manner, the standard rule of stochastic derivation associated with stochastic differential equations (SDE.) (6_s) when the drift vector field $f(t,x) \in R^n$ and diffusion vector fields $g_j(t,x) \in R^n$, $j \in \{1,\ldots,m\}$ are not bounded with respect to $(t,x) \in [0,T] \times R^n$. More precisely, we are given continuous functions

$$f(t,x), g_j(t,x) : [0,T] \times R^n \to R^n, j = 1, \ldots, m$$

such that

(1) $||h(t,x'') - h(t,x')|| \leqslant L||x'' - x'||$, for any $x', x'' \in R^n, t \in [0,T]$ where $L > 0$ is a constant and h stands for f or g_j, $j \in \{1, \ldots, m\}$

(2) $g_j \in C_b^{1,2}([0,T] \times R^n)$ i.e. $\partial_t g_j, \partial_x g_j, \partial_{tx}^2 g_j$ and $\partial_{x^2}^2 g_j$, $j \in \{1, \ldots, m\}$ are continuous and bounded functions

Consider the following system of SDE

(3) $dx = f(t,x)dt + \sum_{j=1}^{m} g_j(t,x) \circ dw^j(t), x(0) = x_0,\ t \in [0,T]$ where Fisk-Stratonovich integral "0" is related to It ô stochatisc integral "." by

$$g_j(t,x) \circ dw^j(t) \overset{def}{=} g_j(t,x) \bullet dw^j(t) + \frac{1}{2}[d_x g_j(t,x)]g_j(t,x)dt$$

Assuming that $\{f, g_j, j = 1, \ldots, m\}$, fulfil the hypotheses (1), (2) then there is a unique solution $x(t) : [0,T] \to R^n$ which is a continuous and F_t-adapted process satisfying the corresponding integral equation

(4) $x(t) = x_0 + \int_0^t f(s,x(s))ds + \sum_{j=1}^{m} \int_0^t g_j(s,x(s)) \circ dw^j(s), t \in [0,T]$

(see A. Friedman). Here $w(t) = (w^1(t), \ldots, w^m(t) : [0,T] \to R^m$ is the standard Wiener process over the filtered probability space $\{\Omega, \{\mathcal{F}_t\} \uparrow \subseteq \mathcal{F}, P\}$ and $\{\mathcal{F}_t, t \in [0,T]\} \subseteq \mathcal{F}$ is the corresponding increasing family of σ-algebras. Consider the exist ball time

(5) $\tau \overset{def}{=} \inf\{t \in [0,T] : x(t) \notin B(x_0, \rho)\}$ associated with the unique solution of (4) and the ball centered at $x_0 \in R^n$ with radius $\rho > 0$.

By definition, each set $\{\omega : \tau \geqslant t\}$ belongs to $\mathcal{F}_t, t \in [0,T]$ and the characteristic function $\chi_\tau(t) : [0,T] \to \{0,1\}, \chi_\tau(t) = 1$ for $\tau \geqslant t$, $\chi_\tau(t) = 0$ for $\tau < t$, is an $\mathcal{F}_t$-adapted process

Theorem 4.3.1. *(stochastic rule of derivation)*

Assume that the hypotheses (1) and (2) are fulfilled and let $\{x(t) : t \in [0,T]\}$ be the unique solution of (4).

Define a stopping time $\tau : \Omega \to [0,T]$ as in (5) and consider a smooth scalar function $\varphi \in C^{1,3}([0,T] \times R^n)$. Then the following integral equation is valid

(6) $\varphi(t \wedge \tau,\, x(t \wedge \tau)) = \varphi(0, x_0) + \int_0^{t\wedge\tau} [\partial_s \varphi(s, x(s)) + < \partial_x \varphi(s, x(s)),$
$f(s, x(s)) >]ds + \sum_{j=1}^m \int_0^{t\wedge\tau} < \partial_x \varphi(s, x(s)) g_j(s, x(s)) > \circ\, dw^j(s),$
$t \in [0, T]$

where the Fisk-Stratonovich integral "0" is defined by

$h_j(s, x(s)) \circ dw^j(s) = h_j(s, x(s)).dw^j(s) + \frac{1}{2}[\partial_x h_j(s, x(s))] g_j(s, x(s))]ds$

using Itô stochastic integral ".".

Proof. Denote $\alpha(t) = \varphi(t, x(t)) \in R$, $z(t) = \mathrm{col}(\alpha(t), x(t) \in R^n$, $t \in [0, T]$ where $\{x(t), t \in [0, T]\}$ is the unique solution of (4) and $\varphi \in C^{1,3}([0, T] \times R^n)$ is fixed. Define new vector fields $\widehat{f}(t, x) = \mathrm{col}(h_0(t, x), f(t, x) \in R^{n+1}$,

(7) $\widehat{g}_j(t, x) = \mathrm{col}(h_j(t, x), g_j(t, x)) \in R^{n+1}$, $j \in \{1, \ldots, m\}$,

where

$h_0(t, x) \stackrel{def}{=} \partial_t \varphi(t, x) + < \partial_x \varphi(t, x), f(t, x) >$

and

$h_j(t, x) = < \partial_x \varphi(t, x), g_j(t, x) >, (t, x) \in [0, T] \times R^n$, $j \in \{1 \ldots, m\}$, $z = \mathrm{col}(\alpha, x) \in R^{n+1}$.

Let $\gamma(x) : R^n \to [0, 1]$ be a smooth function ($\gamma \in C^\infty(R^n)$) such that $\gamma(x) = 1$ if $x \in B(x_0, \rho)$, $\gamma(x) = 0$ if $x \in R^n \setminus B(x_0, 2\rho)$ and $0 \leqslant \gamma(x) \leqslant 1$ for any $x \in B(x_0, 2\rho) \setminus B(x_0, \rho)$, where $\rho > 0$ is fixed arbitrarily. Multiplying $\widehat{f}$ and $\widehat{g_j}$ by $\gamma \in C^\infty(R^n)$ we get

(8) $f^\rho(t, x) \stackrel{def}{=} \gamma(x)\, \widehat{f}(t, x)$, $g_j^\rho(t, x) = \gamma(x) \widehat{g}_j(t, x)$, $j = 1, \ldots, m$, as smooth functions satisfying the hypothesis of Theorem 4.2.1 and denote $\{z^\rho(t), t \in [0, T]\}$ the unique solution fulfilling the foloowing system of SDE. $(z^\rho(t) = (\alpha^\rho(t), x^\rho(t))$

(9) $z^\rho(t) = z_0 + \displaystyle\int_0^t f^\rho(s, x^\rho(s))ds + \sum_{j=1}^m \int_0^t g_j^\rho(s, x^\rho(s)) \circ dw^j(s)$, $t \in [0, T]$, where $z_0 = \mathrm{col}(\varphi(0, x_0), x_0) \in R^{n+1}$ and
$g_j^\rho(s, x^\rho(s)) \circ dw^j)(s) = g_j^\rho(s, x^\rho(s)) \bullet dw^j(s) + \frac{1}{2}[\partial_x g_j^\rho(s, x^\rho(s))] g_j^\rho(s, x^\rho(s))ds.$

In particular, for $t \in [0, \tau]$ and using the characteristic function $\chi_\tau(t)$, $t \in [0, T]$, (see (5)) we rewrite (9) as

(10) $z^\rho(t \wedge \tau) = z_0 + \displaystyle\int_0^t \chi_\tau(s) \widehat{f}(s, x(s))ds + \sum_{j=1}^m \int_0^t \chi_\partial(s) \widehat{g}_j(s, x(s)) \circ dw(s)$ for any

$t \in [0, T]$, where $z^\rho(t \wedge \tau) = (\alpha^\rho(t \wedge \tau), x(t \wedge \tau))$ and $\widehat{g}_j(\widehat{f})$ are defined in (7).

Using Remark 4.2.3, we get

(11)
$$\lim_{\varepsilon \to 0} E||z_\varepsilon^\rho(t) - z^\rho(t \wedge \tau)|| = 0, \text{ for each } t \in [0, T]$$

Here $z_\varepsilon^\rho \stackrel{def}{=} z_\varepsilon^\rho(t\wedge\tau), t\in[0,T]$, verifies the following system of ODE.

$$\begin{cases} \dfrac{dz}{dt} = \chi_\tau(t) f^\rho(t,x) + \displaystyle\sum_{j=1}^m \chi_\tau(t) g_j(t,x)\dfrac{dv_\varepsilon^j(t)}{dt}, t\in[0,T] \\ z(0) = z_0 = (\varphi(0,x_0),x_0) \end{cases} \tag{12}$$

where the vector fields $f^\rho, g_j^\rho, f\in\{1,\dots m\}$ are defined in (8) and fulfil the hypothesis of Theorem 4.2.1. By definition, $z_\varepsilon^\rho(t) = (\alpha_\varepsilon^\rho(t), x_\varepsilon^\rho(t), t\in[0,T]$, and (12) can be rewritten as follows

$$\begin{cases} \dfrac{d\alpha_\varepsilon^\rho(t)}{dt} = \chi_\tau(t)\gamma(x_\varepsilon^\rho(t))\Big[h_0(t,x_\varepsilon^\rho(t)) + \displaystyle\sum_{j=1}^m h_j(t,x_\varepsilon^\rho(t))\dfrac{dv_\varepsilon^j(t)}{dt}\Big] \\ \dfrac{dx_\varepsilon^\rho(t)}{dt} = \chi_\tau(t)\gamma(x_\varepsilon^\rho(t))\Big[f(t,x_\varepsilon^\rho(t)) + \displaystyle\sum_{j=1}^m g_j(t,x_\varepsilon^\rho(t))\dfrac{dv_\varepsilon^j(t)}{dt}\Big] \end{cases} \tag{13}$$

$$\alpha_\varepsilon^\rho(0) = \varphi(0,x_0),\ x_\varepsilon^\rho(0) = x_0,\ t\in[0,T],$$

where the scalar functions $h_i, i\in\{0,1,\dots,m\}$, are given in (7).

In a similar way, write $z^\rho(t) = (\alpha^\rho(t), x^\rho(t)), t\in[0,T]$ and using (10) we get that $\alpha^\rho(t\wedge\tau) = \widehat{\alpha}^\rho(t)$ and $x^\rho(t\wedge\tau) = \widehat{x}^\rho(t)$, $t\in[0,T]$ fulfil the following system of SDE.

(14)
$$\begin{cases} \widehat{\alpha}^\rho(t) = \varphi(0,x_0) + \displaystyle\int_0^t \chi_\tau(s) h_0(s,x(s))ds + \sum_{j=1}^m, \int_0^t \chi_\tau(s) h_j(s,x(s))\circ dw^j(s) \\ x(t\wedge\tau) = \widehat{x}^\rho(t) = x_0 + \displaystyle\int_0^t \chi_\tau(s) f(s,x(s))ds + \sum_{j=1}^m \int_0^t \chi_\tau(s) g_j(s,x(s))\circ dw^j(s) \end{cases}$$

Notice that $\alpha_\varepsilon^\rho(t) = \varphi(t\wedge\tau, x_\varepsilon^\rho(t))$ and $x_\varepsilon^\rho(t), t\in[0,T]$, $\varepsilon>0$, are bounded and convergent to $\widehat{\alpha}^\rho(t), x(t\wedge\tau) = \widehat{x}^\rho(t)$, correspondigly, for each $t\in[0,T]$ (see (11)), when $\varepsilon\to 0$.

As a consequence

$$\begin{aligned} &\varphi(t\wedge\tau, x(t\wedge\tau)) = \lim_{\varepsilon\to 0}\varphi(t\wedge\tau, x_\varepsilon^\rho(t)) = \lim_{\varepsilon\to 0}\alpha_\varepsilon^\rho(t) = \widehat{\alpha}^\rho(t) = \\ &= \varphi(0,x_0) + \int_0^{t\wedge\tau} h_0(s,x(s))ds + \sum_{j=1}^m \int_0^{t\wedge\tau} h_j(s,x(s))\circ dw^j(s),\ t\in[0,T] \end{aligned} \tag{15}$$

and the proof of the conclusion (6) is complete. □

Comment on stochastic rule of derivation

The stochastic rule of derivation is based on the hypothesis (2) which involves higher differentiability properties of the diffusion vector fields

$$g_j \in C_b^{1,2}([0,T] \times R^n; R^n), j \in \{1, \ldots, m\}$$

On the other hand, using a stopping time $\tau : \Omega \to [0,T]$, the right hand side of the equation (6) is a semimartingale without imposing any growth condition on the test function $\varphi \in C^{1,3}([0,T] \times R^n)$. The standard stochastic rule of derivation does not contain a stopping time and it can be accomplished assuming the following growth condition

(16) $|\partial_t \varphi(t,x)|, |\partial_{x_i} \varphi(t,x)|, |\partial^2_{x_i x_j} \varphi(t,x)| \leqslant k(1 + ||x||^p)$, $i, j \in \{1, \ldots, n\}$, $x \in R^n$, where $p \geqslant 1$ (natural) and $k > 0$ are fixed. Adding condition (16) to the hypotheses (1) and (2) of Theorem 4.3.1, and using a sequence of stopping times $\tau_\rho : \Omega \to [0,T]$, $\lim\limits_{\rho \to \infty} \tau_\rho = T$, we get the following stochastic rule of derivation

$$(17)\quad \begin{aligned} \varphi(t, x(t)) = \varphi(0, x_0) + \int_0^t [\partial_s \varphi(s, x(s)) + < \partial_x \varphi(s, x(s)), f(s, x(s)) >] ds + \\ + \sum_{j=1}^m \int_0^t < \partial_x \varphi(s, x(s)), g_j(s, x(s)) > \circ dw^j(s),\ t \in [0,T] \end{aligned}$$

.

4.4 Appendix

I. Two problems for stochastic flows asociated with nonlinear parabolic equations

(I. Molnar , C. Varsan, Functionals associated with gradient stochastic flows and nonlinear parabolic equations, preprint IMAR 12/2009)

4.4.1 Introduction

Consider that $\{\widehat{x}_\varphi(t;\lambda) : t \in [0,T]\}$ is the unique solution of SDE driven by complete vector fields $f \in (\mathcal{C}_b \cap \mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n;\mathbb{R}^n)$ and $g \in (\mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n;\mathbb{R}^n)$,

$$\begin{cases} d_t\widehat{x} = \varphi(\lambda)f(\widehat{x})dt + g(\widehat{x}) \circ dw(t),\ t \in [0,T],\ x \in \mathbb{R}^n, \\ \widehat{x}(0) = \lambda \in \mathbb{R}^n, \end{cases} \tag{4.34}$$

where $\varphi \in (\mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n)$ and $w(t) \in \mathbb{R}$ is a scalar Wiener process over a complete filtered probability space $\{\Omega, \mathcal{F} \supset \{\mathcal{F}_t\}, P\}$. We recall that Fisk-Stratonovich integral "$\circ$" in (4.34) is computed by

$$g(x) \circ dw(t) = g(x) \cdot dw(t) + \frac{1}{2}\partial_x g(x) \cdot g(x)dt,$$

using Ito stochastic integral "$\cdot$".

We are going to introduce some nonlinear SPDE or PDE of parabolic type which describe the evolution of stochastic functionals $u(t,x) := h(\psi(t,x))$, or $S(t,x) := Eh(\widehat{x}_\psi(T;t,x))$, $t \in [0,T]$, $x \in \mathbb{R}^n$, for a fixed $h \in (\mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n)$. Here $\{\lambda = \psi(t,x) : t \in [0,T], x \in \mathbb{R}^n\}$ is the unique solution satisfying integral equations

$$\widehat{x}_\varphi(t;\lambda) = x \in \mathbb{R}^n,\ t \in [0,T]. \tag{4.35}$$

The evolution of $\{S(t,x) : t \in [0,T], x \in R^n\}$ will be defined by some nonlinear backward parabolic equation considering that $\{\widehat{x}_\psi(s;t,x) : s \in [t,T], x \in R^n\}$ is the unique solution of SDE

$$\begin{cases} d_s\widehat{x} = \varphi(\psi(t,x))f(\widehat{x})ds + g(\widehat{x}) \circ dw(s),\ s \in [t,T], \\ \widehat{x}(t) = x \in \mathbb{R}^n. \end{cases}$$

4.4.2 Some problems and their solutions

Problem (P1). Assume that g and f commute using Lie bracket, i.e.

$$[g,f](x) = 0,\ x \in \mathbb{R}^n, \tag{4.36}$$

where $[g,f](x) := [\partial_x g(x)]f(x) - [\partial_x f(x)]g(x)$,

$$TVK = \rho \in [0,1), \tag{4.37}$$

where $V = \sup\{|\partial_x\varphi(x)| : x \in \mathbb{R}\}^n$ and $K = \sup\{|f(x)|; x \in \mathbb{R}^n\}$.

Under the hypotheses (4.36) and (4.37), find the nonlinear SPDE of parabolic type satisfied by $\{u(t,x) = h(\psi(t,x)) :\in [0,T], x \in \mathbb{R}^n\}$, $h \in (\mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n)$, where $\{\lambda = \psi(t,x) \in \mathbb{R}^n : t \in [0,T], x \in \mathbb{R}^n\}$ is the unique continuous and $\mathcal{F}_t$-adapted solution of the integral equation (4.35).

Problem (P2). Using $\{\lambda = \psi(t,x)\}$ found in (P1), describe the evolution of a functional $S(t,x) := Eh(\widehat{x}_\psi(T;t,x))$ using backward parabolic equations, where $\{\widehat{x}_\psi(s;t,x) : s \in [t,T]\}$ is the unique solution of SDE

$$\begin{cases} d_s\widehat{x} = \varphi(\psi(t,x))f(\widehat{x})ds + g(\widehat{x}) \circ dw(s),\ s \in [t,T] \\ \widehat{x}(t) = x \in \mathbb{R}^n. \end{cases} \tag{4.38}$$

4.4.3 Solution for the Problem (P1)

Remark 4.4.1. *Under the hypotheses* (4.36) *and* (4.37) *of (P1), the unique solution of integral equations* (4.35) *will be found as a composition*

$$\psi(t,x) = \widehat{\psi}(t, \widehat{z}(t,x)), \tag{4.39}$$

where $\widehat{z}(t,x) := G(-w(t))[x]$ *and* $\lambda = \widehat{\psi}(t,z)$, $t \in [0,T]$, $z \in \mathbb{R}^n$, *is the unique deterministic solution satisfying integral equations*

$$\lambda = F(-\theta(t;\lambda))[z] =: \widehat{V}(t,z;\lambda),\ t \in [0,T],\ z \in \mathbb{R}^n. \tag{4.40}$$

Here $F(\sigma)[z]$ *and* $G(\tau)[z]$, $\sigma, \tau \in \mathbb{R}$, *are the global flows generated by complete vector fields* f *and* g *correspondingly, and* $\theta(t;\lambda) = t\varphi(\lambda)$. *The unique solution of* (4.40) *is constructed in the following*

Lemma 4.4.2. *Assume that* (4.37) *is fulfilled. Then there exists a unique smooth deterministic mapping* $\{\lambda = \widehat{\psi}(t,z) : t \in [0,T], x \in \mathbb{R}^n\}$ *solving integral equations* (4.40) *such that*

$$\begin{cases} F(\theta(t;\widehat{\psi}(t,z)))[\widehat{\psi}(t,z)] = z \in \mathbb{R}^n,\ t \in [0,T], \\ |\widehat{\psi}(t,z) - z| \leqslant R(T,z) := \dfrac{r(T,z)}{1-\rho},\ t \in [0,T],\ \text{where } r(T,z) = TK|\varphi(z)|, \end{cases} \tag{4.41}$$

$$\begin{cases} \partial_t\widehat{\psi}(t,z) + \partial_z\widehat{\psi}(t,z)f(z)\varphi(\widehat{\psi}(t,z)) = 0,\ t \in [0,T],\ x \in \mathbb{R}^n, \\ \widehat{\psi}(0,z) = z \in \mathbb{R}^n. \end{cases} \tag{4.42}$$

Proof. The mapping $\widehat{V}(t,z;\lambda)$ (see (4.40)) is a contractive application with respect to $\lambda \in \mathbb{R}^n$, uniformly of $(t,z) \in [0,T] \times \mathbb{R}^n$ which allows us to get the unique solution of (4.40) using a standard procedure (Banach theorem). By a direct computation, we

get

$$|\partial_\lambda \widehat{V}(t,z;\lambda)| = |f(\widehat{V}(t,z;\lambda))\partial_\lambda\theta(t;\lambda)| \leqslant TVK = \rho \in [0,1), \tag{4.43}$$

for any $t \in [0,T]$, $z \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^n$, where $\partial_\lambda\theta(t;\lambda)$ is a row vector. The corresponding convergent sequence $\{\lambda_k(t,z) : t \in [0,T], z \in \mathbb{R}^n\}_{k\geqslant 0}$ is constructed fulfilling

$$\lambda_0(t,z) = z,\ \lambda_{k+1}(t,z) = \widehat{V}(t,z;\lambda_k(t,z)),\ t \geqslant 0, \tag{4.44}$$

$$\begin{cases} |\lambda_{k+1}(t,z) - \lambda_k(t,z)| \leqslant \rho^k|\lambda_1(t,z) - \lambda_0(t,z)|,\ k \geqslant 0, \\ |\lambda_1(t,z) - \lambda_0(t,z)| \leqslant |\widehat{V}(t,z;z) - z| \leqslant TK|\varphi(z)| =: r(T,z). \end{cases} \tag{4.45}$$

Using (4.45) we obtain that $\{\lambda_k(t,z)\}_{k\geqslant 0}$ is convergent and

$$\widehat{\psi}(t,z) = \lim_{k\to\infty} \lambda_k(t,z),\ |\widehat{\psi}(t,z) - z| \leqslant \frac{r(T,z)}{1-\rho} =: R(T,z),\ t \in [0,T]. \tag{4.46}$$

Passing $k \to \infty$ into (4.44) and using (4.46) we get the first conclusion (4.41). On the other hand, notice that $\{\widehat{V}(t,z;\lambda) : t \in [0,T], z \in \mathbb{R}^n\}$ of (4.40) fulfils

$$\widehat{V}(t,\widehat{y}(t,\lambda);\lambda) = \lambda,\ t \in [0,T],\ \text{where } \widehat{y}(t,\lambda) = F(\theta(t;\lambda))[\lambda]. \tag{4.47}$$

This shows that all the components of $\widehat{V}(t,z;\lambda) \in \mathbb{R}^n$ are first integrals associated with the vector field $f_\lambda(z) = \varphi(\lambda)f(z)$, $z \in \mathbb{R}^n$, for each $\lambda \in \mathbb{R}^n$, i.e.

$$\partial_t\widehat{V}(t,\widehat{y}(t,\lambda);\lambda) + [\partial_z\widehat{V}(t,\widehat{y}(t,\lambda);\lambda)]f(\widehat{y}(t,\lambda))\varphi(\lambda) = 0,\ t \in [0,T] \tag{4.48}$$

is valid for each $\lambda \in \mathbb{R}^n$. In particular, for $\lambda = \widehat{\psi(t,z)}$ we get $\widehat{y}(t,\widehat{\psi}(t,z)) = z$ and (4.48) becomes (H-J)-equation

$$\partial_t\widehat{V}(t,z;\widehat{\psi}(t,z)) + [\partial_z\widehat{V}(t,z;\widehat{\psi}(t,z)]f(z)\varphi(\widehat{\psi}(t,z)) = 0,\ t \in [0,T],\ z \in \mathbb{R}^n. \tag{4.49}$$

Combining (4.40) and (4.49), by direct computation, we convince ourselves that $\lambda = \widehat{\psi}(t,z)$ fulfils the following nonlinear (H-J)-equation (see (4.42))

$$\begin{cases} \partial_t\widehat{\psi}(t,z) + [\partial_z\widehat{\psi}(t,z)]f(z)\varphi(\widehat{\psi}(t,z)) = 0,\ t \in [0,T],\ z \in \mathbb{R}^n, \\ \widehat{\psi}(0,z) = z \in \mathbb{R}^n, \end{cases} \tag{4.50}$$

and the proof is complete. □

Remark 4.4.3. *Under the hypothesis* (4.36)*, the* $\{\widehat{x}_\varphi(t;\lambda) : t \in [0,T], \lambda \in \mathbb{R}^n\}$ *generated by SDE* (4.34) *can be represented as follows*

$$\widehat{x}_\varphi(t;\lambda) = G(w(t)) \circ F(\theta(t;\lambda))[\lambda] = H(t,w(t);\lambda),\ t \in [0,T],\ \lambda \in \mathbb{R}^n \tag{4.51}$$

where $\theta(t;\lambda) = t\varphi(\lambda)$.

Lemma 4.4.4. *Assume that* (4.36) *and* (4.37) *are satisfied and consider* $\{\lambda = \widehat{\psi}(t,z) : t \in [0,T], z \in \mathbb{R}^n\}$ *found in Lemma 4.4.2. Then the stochastic flow gen-*

erated by SDE (4.34) *fulfils*

$$\{\widehat{x}_{\varphi}(t;\lambda) : t \in [0,T], \lambda \in \mathbb{R}^n\} \text{ can be represented as in } (4.51), \tag{4.52}$$

$$\begin{aligned} &\psi(t,x) = \widehat{\psi}(t,\widehat{z}(t,x)) \text{ is the unique solution of integral equations } (4.35), \\ &\text{where } \widehat{z}(t,x) = G(-w(t))[x]. \end{aligned} \tag{4.53}$$

Proof. Using the hypothesis (4.36), we see easily that

$$y(\theta,\sigma)[\lambda] := G(\sigma) \circ F(\theta)[\lambda],\ \theta,\sigma \in \mathbb{R},\ \lambda \in \mathbb{R}^n \tag{4.54}$$

is the unique solution of the gradient system

$$\begin{cases} \partial_\theta y(\theta,\sigma)[\lambda] = f(y(\theta,\sigma)[\lambda]),\ \partial_\sigma y(\theta,\sigma)[\lambda] = g(y(\theta,\sigma)[\lambda]), \\ y(0,0)[\lambda] = \lambda \end{cases} \tag{4.55}$$

Applying the standard rule of stochastic derivation associated with the smooth mapping $\varphi(\theta,\sigma) := y(\theta,\sigma)[\lambda]$ and the continuous process $\theta = \theta(t;\lambda) = t\varphi(\lambda)$, $\sigma = w(t)$, we get that $\widehat{y}_{\varphi}(t;\lambda) = y(\theta(t;\lambda), w(t))$, $t \in [0,T]$, fulfils SDE (4.34), i.e.

$$\begin{cases} d_t\widehat{y}_{\varphi}(t;\lambda) = \varphi(\lambda) f(\widehat{y}_{\varphi}(t;x))dt + g(\widehat{y}_{\varphi}(t;\lambda)) \circ dw(t),\ t \in [0,T], \\ \widehat{y}_{\varphi}(0;\lambda) = \lambda. \end{cases} \tag{4.56}$$

On the other hand, the unicity of the solution satisfying (4.34) lead us to the conclusion that $\widehat{x}_{\varphi}(t;\lambda) = \widehat{y}_{\varphi}(t;\lambda)$, $t \in [0,T]$, and (4.52) is proved. The conclusion (4.53) is a direct consequence of (4.52) combined with $\{\lambda = \widehat{\psi}(t,z) : t \in [0,T], z \in \mathbb{R}^n\}$ is the solution defined in Lemma 4.4.2. The proof is complete. □

Lemma 4.4.5. *Under the hypotheses in Lemma 4.4.4, consider the continuous and* $\mathcal{F}_t$*-adapted process* $\widehat{z}(t,x) = G(-w(t))[x]$, $t \in [0,T]$, $x \in \mathbb{R}^n$. *Then the following SPDE of parabolic type is valid*

$$\begin{cases} d_t\widehat{z}(t,x) + \partial_x\widehat{z}(t,x)g(x)\widehat{\circ}dw(t) = 0,\ t \in [0,T], x\, x \in \mathbb{R}^n, \\ \widehat{z}(0,x) = x \end{cases} \tag{4.57}$$

where the "$\widehat{\circ}$*" is computed by*

$$h(t,x)\widehat{\circ}dw(t) = h(t,x) \cdot dw(t) - \frac{1}{2}\partial_x h(t,x)g(x)dt,$$

using Ito "$\cdot$*".*

Proof. The conclusion (4.57) is a direct consequence of applying standard rule of stochastic derivation associated with $\sigma = w(t)$ and smooth deterministic mapping $H(\sigma)[x] := G(-\sigma)[x]$. In this respect, using $H(\sigma) \circ G(\sigma)[\lambda] = \lambda \in \mathbb{R}^n$ for any

$x = G(\sigma)[\lambda]$, we get

$$\begin{cases} \partial_\sigma\{H(\sigma)[x]\} = -\partial_x\{H(\sigma)[x]\} \cdot g(x),\, \sigma \in \mathbb{R},\, x \in \mathbb{R}^n, \\ \partial_\sigma^2\{H(\sigma)[x]\} = \partial_\sigma\{\partial_\sigma\{H(\sigma)[x]\}\} = \partial_\sigma\{-\partial_x\{H(\sigma)[x]\} \cdot g(x)\} \\ \qquad\qquad = \partial_x\{\partial_x\{H(\sigma)[x]\} \cdot g(x)\} \cdot g(x),\, \sigma \in \mathbb{R},\, x \in \mathbb{R}^n. \end{cases} \tag{4.58}$$

The standard rule of stochastic derivation lead us to SDE

$$d_t\widehat{z}(t,x) = \partial_\sigma\{H(\sigma)[x]\}_{\sigma=w(t)} \cdot dw(t) + \frac{1}{2}\partial_\sigma^2\{H(\sigma)[x]\}_{\sigma=w(t)}dt,\, t \in [0,T], \tag{4.59}$$

and rewritting the right hand side of (4.59) (see (4.58)) we get SPDE of parabolic type given in (4.57). The proof is complete. □

Lemma 4.4.6. *Assume the hypotheses* (4.36) *and* (4.37) *are fulfilled and consider* $\{\lambda = \psi(t,x) : t \in [0,T], x \in \mathbb{R}^n\}$ *defined in Lemma* (4.4.4)*. Then* $u(t,x) := h(\psi(t,x))$, $t \in [0,T]$, $x \in \mathbb{R}^n$, $h \in (\mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n)$, *satisfies the following nonlinear SPDE of parabolic type*

$$\begin{cases} d_t u(t,x) + \langle \partial_x u(t,x), f(x)\rangle \varphi(\psi(t,x))dt + \langle \partial_x u(t,x), g(x)\rangle \widehat{\circ} dw(t) = 0 \\ u(0,x) = h(x),\, t \in [0,T],\, x \in \mathbb{R}^n, \end{cases} \tag{4.60}$$

where the "$\widehat{\circ}$*" is computed by*

$$h(t,x)\widehat{\circ}dw(t) = h(t,x) \cdot dw(t) - \frac{1}{2}\partial_x h(t,x)g(x)dt.$$

Proof. By definition (see Lemma (4.4.4)), $\psi(t,x) = \widehat{\psi}(t,\widehat{z}(t,x))$, $t \in [0,T]$, where $\widehat{z}(,x) = G(-w(t))[x]$ and $\{\widehat{\psi}(t,z) \in \mathbb{R}^n : t \in [0,T], z \in \mathbb{R}^n\}$ satisfies nonlinear (H-J)-equations (4.42) of Lemma 4.4.2. In addition $\{\widehat{z}(t,x) \in \mathbb{R}^n : t \in [0,T], x \in \mathbb{R}^n\}$ fulfils SPDE (4.57) in Lemma 4.4.5, i.e.

$$d_t\widehat{z}(t,x) + \partial_x\widehat{z}(t,x)\widehat{\circ}dw(t) = 0,\, t \in [0,T],\, x \in \mathbb{R}^n. \tag{4.61}$$

Applying the standard rule of stochastic derivation associated with the smooth mapping $\{\lambda = \widehat{\psi}(t,z) : t \in [0,T], z \in \mathbb{R}^n\}$ and stochastic process $\widehat{z}(t,x) := G(-w(t))[x] =: H(w(t))[x]$, $t \in [0,T]$, we get the following nonlinear SPDE

$$\begin{cases} d_t\psi(t,x) + \partial_x\psi(t,x)f(x)\varphi(\psi(t,x))dt + \partial_x\psi(t,x)g(x)\widehat{\circ}dw(t) = 0, \\ \psi(0,x) = x,\, t \in [0,T]. \end{cases} \tag{4.62}$$

In addition, the functional $u(t,x) = h(\psi(t,x))$ can be rewritten $u(t,x) = \widehat{u}(t,\widehat{z}(t,x))$, where $\widehat{u}(t,z) := h(\widehat{\psi}(t,z))$ is a smooth satisfying nonlinear (H-J)-equations (see (4.42) of Lemma 4.4.2)

$$\begin{cases} \partial_t\widehat{u}(t,z) + \langle \partial_z\widehat{u}(t,z), f(z)\rangle \varphi(\widehat{\psi}(t,z)) = 0,\, t \in [0,T],\, z \in \mathbb{R}^n, \\ \widehat{u}(0,z) = h(z). \end{cases} \tag{4.63}$$

Using (4.61)and (4.63) we obtain SDPE fulfilled by $\{u(t,x)\}$,

$$\begin{cases} d_t u(t,x) + \langle \partial_z \widehat{u}(t,\widehat{z}(t,x)), f(\widehat{z}(t,x))\rangle \varphi(\psi(t,x))dt + \langle \partial_x u(t,x), g(x)\rangle \widehat{\circ} dw(t) = 0, \\ u(0,x) = h(x),\ t \in [0,T],\ x \in \mathbb{R}^n. \end{cases} \tag{4.64}$$

The hypothesis (4.36) allows us to write

$$\begin{aligned} \langle \partial_z \widehat{u}(t,\widehat{z}(t,x)), f(\widehat{z}(t,x))\rangle &= \partial_z \widehat{u}(t,\widehat{z}(t,x))[\partial_x \widehat{z}(t,x)][\partial_x \widehat{z}(t,x)]^{-1} f(\widehat{z}(t,x)) \\ &= \langle \partial_x u(t,x), f(x)\rangle,\ t \in [0,T],\ x \in \mathbb{R}^n, \end{aligned} \tag{4.65}$$

and using (4.65) into (4.64) we get the conclusion (4.60),

$$\begin{cases} \partial_t u(t,x) + \langle \partial_x u(t,x), f(x)\rangle \varphi(\psi(t,x))dt + \langle \partial_x u(t,x), g(x)\rangle \widehat{\circ} dw(t) = 0, \\ u(0,x) = h(x),\ t \in [0,T], x \in \mathbb{R}^n, \end{cases} \tag{4.66}$$

where the "$\widehat{\circ}$" is computed by

$$h(t,x)\widehat{\circ} dw(t) = -\frac{1}{2}\partial_x h(t,x) g(x)dt + h(t,x) \cdot dw(t), \tag{4.67}$$

using Ito integral "$\cdot$". The proof is complete. □

Remark 4.4.7. *The complete solution of Problem (P1) is contained in Lemmas 4.4.2–4.4.6. We shall rewrite them as a theorem.*

Theorem 4.4.8. *Assume that the vector fields* $f \in (\mathcal{C}_b \cap \mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n;\mathbb{R}^n)$*,* $g \in (\mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n;\mathbb{R}^n)$*, and scalar function* $\varphi \in (\mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n)$ *fulfil the hypotheses* (4.36) *and* (4.37)*. Consider the continuous and* $\mathcal{F}_t$*-* $\{\lambda = \psi(t, x \in \mathbb{R}^n) : t \in [0,T], x \in \mathbb{R}^n\}$ *satisfying integral equations* (4.35)*. Then* $u(t,x) := h(\psi(t,x))$*,* $t \in [0,T]$*,* $x \in \mathbb{R}^n$*, fulfils nonlinear SPDE of parabolic type* (4.60) *(see Lemma 4.4.6), for each* $h \in (\mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n)$*.*

4.4.4 Solution for the Problem (P2)

Using the same notations as in subsection 4.4.3, we consider the unique solution $\{\widehat{x}_\psi(s;t,x) : s \in [t,T]\}$ satisfying SDE (4.38) for each $0 \leqslant t < T$ and $x \in \mathbb{R}^n$. As far as SDE (4.38) is a , the evolution of a functional $S(t,x) := Eh(\widehat{x}_\psi(T;t,x))$, $t \in [0,T]$, $x \in \mathbb{R}^n$, $h \in (\mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n)$, will be described using the pathwise representation of the conditional mean values functional

$$v(t,x) = E\{h(\widehat{x}_\psi(T;t,x)) \mid \psi(t,x)\},\ 0 \leqslant t < T,\ x \in \mathbb{R}^n. \tag{4.68}$$

Assuming the hypotheses (4.36) and (4.37) we may and do write the following integral representation

$$\widehat{x}_\psi(T;t,x) = G(w(T) - w(t)) \circ F[(T-t)\varphi(\psi(t,x))][x],\ 0 \leqslant t < T,\ x \in \mathbb{R}^n, \tag{4.69}$$

for a solution of SDE (4.38), where $G(\sigma)[z]$ and $F(\tau)[z]$, $\sigma, \tau \in \mathbb{R}$, $z \in \mathbb{R}^n$, are the global flows generated by $g, f \in (\mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n; \mathbb{R}^n)$. The right side hand of (4.69) is a continuous mapping of the two independent random variables, $z_1 = [w(T) - w(t)] \in \mathbb{R}$ and $z_2 = \psi(t, x) \in \mathbb{R}^n$ ($\mathcal{F}_t$-measurable) for each $0 \leqslant t < T$, $x \in \mathbb{R}^n$. A direct consequence of this remark is to use a

$$y(t, x; \lambda) = G(w(T) - w(t)) \circ F[(T - t)\varphi(\lambda)][x],\ 0 \leqslant t < T, \tag{4.70}$$

and to compute the conditional mean values (4.68) by

$$v(t, x) = [Eh(y(t, x; \lambda))](\lambda = \psi(t, x)). \tag{4.71}$$

Here the functional

$$u(t, x; \lambda) := Eh(y(t, x; \lambda)),\ t \in [0, T],\ x \in \mathbb{R}^n, \tag{4.72}$$

satisfies a backward parabolic equation (Kolmogorov's equation) for each $\lambda \in \mathbb{R}^n$ and rewrite (4.71) as follows,

$$v(t, x) = u(t, x; \psi(t, x)),\ 0 \leqslant t < T,\ x \in \mathbb{R}^n. \tag{4.73}$$

In conclusion, the functional $\{S(t, x)\}$ can be written as

$$S(t, x) = E[E\{h(\widehat{x}_\psi(T; t, x)) \mid \psi(t, x)\} = Eu(t, x; \psi(t, x)),\ 0 \leqslant t < T,\ x \in \mathbb{R}^n, \tag{4.74}$$

where $\{u(t, x; \lambda) : t \in [0, T], x \in \mathbb{R}^n\}$ satisfies the corresponding backward parabolic equations with parameter $\lambda \in \mathbb{R}^n$,

$$\begin{cases} \partial_t u(t, x; \lambda) + \langle \partial_x u(t, x; \lambda), f(x, \lambda) \rangle + \dfrac{1}{2} \langle \partial_x^2 u(t, x; \lambda) g(x), g(x) \rangle = 0, \\ u(T, x; \lambda) = h(x),\ f(x, \lambda) := \varphi(\lambda) f(x) + \dfrac{1}{2} [\partial_x g(x)] g(x). \end{cases} \tag{4.75}$$

We conclude these remarks by a theorem.

Theorem 4.4.9. *Assume that the vector fields f, g and the scalar function φ of SDE* (4.38) *fulfil the hypotheses* (4.36) (4.37)*, where the continuous and $\mathcal{F}_t$- $\{\psi(t, x) \in \mathbb{R}^n : t \in [0, T]\}$ is defined in Theorem 4.4.8. Then the evolution of the functional*

$$S(t, x) := Eh(\widehat{x}_\psi(T; t, x)),\ t \in [0, T],\ x \in \mathbb{R}^n,\ h \in (\mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n) \tag{4.76}$$

can be described as in (4.74)*, where $\{u(t, x) : t \in [0, T], x \in \mathbb{R}^n\}$ satisfies linear backward parabolic equations* (4.75) *for each $\lambda \in \mathbb{R}^n$.*

Remark 4.4.10. *Consider the case of several vector fields defining both the drift and diffusion of SDE* (4.34)*, i.e.*

$$\begin{cases} d_t \widehat{x} = [\sum_{i=1}^m \varphi_i(\lambda) f_i(\widehat{x})] dt + \sum_{i=1}^m g_i(\widehat{x}) \circ dw_i(t),\ t \in [0, T], \\ \widehat{x}(0) = \lambda \in \mathbb{R}^n. \end{cases} \tag{4.77}$$

We notice that the analysis presented in Theorems 4.4.8 and 4.4.9 can be extended to this multiple vector fields case (see next section).

4.4.5 Multiple vector fields case

We are given two finite sets of vector fields $\{f_1, \ldots, f_m\} \subset (\mathcal{C}_b \cap \mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n; \mathbb{R}^n)$ and $\{g_1, \ldots, g_m\} \subset (\mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n; \mathbb{R}^n)$ and consider the unique solution $\{\widehat{x}_\varphi(t, \lambda) : t \in [0, T], \lambda \in \mathbb{R}^n\}$ of SDE

$$\begin{cases} d_t\widehat{x} = [\sum_{i=1}^m \varphi_i(\lambda) f_i(\widehat{x})]dt + \sum_{i=1}^m g_i(\widehat{x}) \circ dw_i(t),\ t \in [0, T],\ \widehat{x} \in \mathbb{R}^n, \\ \widehat{x}(0) = \lambda \in \mathbb{R}^n \end{cases} \tag{4.78}$$

where $\varphi = (\varphi_1, \ldots, \varphi_m) \subset (\mathcal{C}_b^1 \cap \mathcal{C}^2)$ are fixed and $w = (w_1(t), \ldots, w_m(t)) \in \mathbb{R}^m$ is a standard Wiener process over a complete filtered probability space $\{\Omega, \mathcal{F} \supset \{\mathcal{F}_t\}, P\}$. Each "$\circ$" in (4.78) is computed by

$$g_i(x) \circ dw_i(t) = g_i(x) \cdot dw_i(t) + \frac{1}{2}[\partial_x g_i(x)] g_i(x) dt, \tag{4.79}$$

using Ito integral "$\cdot$".

Assume that $\{\lambda = \psi(t, x) \in \mathbb{R}^n : t \in [0, T], x \in \mathbb{R}^n\}$ is the unique continuous and $\mathcal{F}_t$-adapted solution satisfying integral equations

$$\widehat{x}_\varphi(t; \lambda) = x \in \mathbb{R}^n,\ t \in [0, T]. \tag{4.80}$$

For each $h \in (\mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n)$, associate stochastic functionals $\{u(t, x) = h(\psi(t, x) : t \in [0, T], x \in \mathbb{R}^n)\}$ and $\{S(t, x) = Eh(\widehat{x}_\psi(T; t, x)) : t \in [0, T], x \in \mathbb{R}^n\}$, where $\{\widehat{x}_\psi(s; t, x) : s \in [t, T], x \in \mathbb{R}^n\}$ satisfies the following SDE

$$\begin{cases} d_s\widehat{x} = [\sum_{i=1}^m \varphi_i(\psi(t, x)) f_i(\widehat{x})]ds + \sum_{i=1}^m g_i(\widehat{x}) \circ dw_i(t),\ s \in [t, T], \\ \widehat{x}(t) = x. \end{cases}$$

Problem (P1). Assume that

$$\begin{cases} M = \{f_1, \ldots, f_m, g_1, \ldots, g_m\} \text{ are muttualy commuting using Lie bracket i.e.} \\ [X_1, X_2](x) = 0 \text{ for any pair } X_1, X_2 \in M \end{cases} \tag{4.81}$$

$$TV_i K_i = \rho_i \in [0, \frac{1}{m}), \tag{4.82}$$

where $V_i := \sup\{|\partial_x \varphi_i(x)| : x \in \mathbb{R}^n\}$ and $K_i = \{|f_i(x)| : x \in \mathbb{R}^n\}$, $i = 1, \ldots, m$.

Under the hypotheses (4.81) and (4.82), find the nonlinear SPDE of parabolic type satisfied by $\{u(t,x) = h(\psi(t,x)), t \in [0,T], x \in \mathbb{R}^n\}$, $h \in (\mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n)$, where $\{\lambda = \psi(t,x) \in \mathbb{R}^n : t \in [0,T], x \in \mathbb{R}^n\}$ is the unique continuous and $\mathcal{F}_t$-adapted solution of integral equations (4.80).

Problem (P2). Using $\{\lambda = \psi(t,x) \in \mathbb{R}^n : t \in [0,T], x \in \mathbb{R}^n\}$ found in (P1), describe the evolution of a functional $S(t,x) = Eh(\widehat{x}_\psi(T;t,x))$ using backward parabolic equations, where $\{\widehat{x}_\psi(s;t,x) : s \in [t,T]\}$ is the unique solution of SDE

$$\begin{cases} d_s\widehat{x} = [\sum_{i=1}^m \varphi_i(\psi(t,x))f_i(\widehat{x})]ds + \sum_{i=1}^m g_i(\widehat{x}) \circ dw_i(s),\ s \in [t,T], \\ \widehat{x}(t) = \widehat{x} \in \mathbb{R}^n. \end{cases} \tag{4.83}$$

4.4.6 Solution for (P1)

Under the hypotheses (4.81) and (4.82), the unique solution of SPDE (4.78) can be represented by

$$\widehat{x}_\varphi(t;\lambda) = G(w(t)) \circ F(\theta(t;\lambda))[\lambda] =: H(t, w(t); \lambda) \tag{4.84}$$

where

$G(\sigma)[z] = G_1(\sigma_1) \circ \cdots \circ G_m(\sigma_m)[z]$, $\sigma = (\sigma_1, \ldots, \sigma_m) \in \mathbb{R}^m$,
$F(\sigma)[z] = F_1(\sigma_1) \circ \cdots \circ F_m(\sigma_m)[z]$, $\theta(t;\lambda) = (t\varphi_1(\lambda), \ldots, t\varphi_m(\lambda)) \in \mathbb{R}^m$ and
$\{(F_i(\sigma_i)[z], G_i(\sigma_i)[z]) : \sigma_i \in \mathbb{R}, z \in \mathbb{R}^n\}$

are the global flows generated by (f_i, g_i), $i \in \{1, \ldots, m\}$.

The arguments for solving (P1) in the case of one pair (f,g) of vector fields (see subsection (4.4.3)) can be used also here and we get the following similar results. Under the representation (4.84), the unique continuous and $\mathcal{F}_t$-adapted solution $\{\lambda = \psi(t,x) : t \in [0,T], x \in \mathbb{R}^n\}$ solving equations

$$\widehat{x}_\varphi(t;\lambda) = x \in \mathbb{R}^n,\ t \in [0,T] \tag{4.85}$$

will be found as a composition

$$\psi(t,x) = \widehat{\psi}(t, \widehat{z}(t,x)),\ \widehat{z}(t,x) := G(-w(t))[x]. \tag{4.86}$$

Here $\lambda = \widehat{\psi}(t, z)$, $t \in [0, T]$, $z \in \mathbb{R}^n$ is the unique solution satisfying

$$\lambda = F(-\theta(t; \lambda))[z] =: \widehat{V}(t, z; \lambda),\ t \in [0, T],\ z \in \mathbb{R}^n. \tag{4.87}$$

Lemma 4.4.11. *Asume that* (4.81) *and* (4.82) *is fulfilled. Then there exists a unique smooth mapping* $\{\lambda = \widehat{\psi}(t, z) : t \in [0, T], z \in \mathbb{R}^n\}$ *solving* (4.87) *such that*

$$\begin{cases} F(\theta(t; \widehat{\psi}(t, z)))[\widehat{\psi}(t, z)] = z \in \mathbb{R}^n,\ t \in [0, T], \\ |\widehat{\psi}(t, z) - z| \leqslant R(T, z) := \dfrac{r(T, z)}{1 - \rho},\ t \in [0, T],\ z \in \mathbb{R}^n, \end{cases} \tag{4.88}$$

where $\rho = \rho_1 + \cdots + \rho_m \in [0, 1)$ *and* $r(T, z) = T \sum_{i=1}^{m} K_i |\varphi_i(z)|$.

In addition, the following nonlinear (H-J)-equation is valid

$$\begin{cases} \partial_t \widehat{\psi}(t, z) + \partial_z \widehat{\psi}(t, z)[\displaystyle\sum_{i=1}^{m} \varphi_i(\widehat{\psi}(t, z)) f_i(z)] = 0,\ t \in [0, T],\ z \in \mathbb{R}^n, \\ \widehat{\psi}(0, z) = z. \end{cases} \tag{4.89}$$

The proof is based on the arguments of Lemma 4.4.2 in subsection 4.4.3.

Lemma 4.4.12. *Assume that* (4.81) *and* (4.82) *are satisfied and consider* $\{\lambda = \widehat{\psi}(t, z) \in \mathbb{R}^n : t \in [0, T], z \in \mathbb{R}^n\}$ *found in Lemma* (4.4.11)*. Then the generated by SDE* (4.78) *fulfils*

$$\{\widehat{x}_\varphi(t; \lambda) : t \in [0, T], \lambda \in \mathbb{R}^n\} \text{ can be represented as in } (4.84), \tag{4.90}$$

$$\begin{aligned} \psi(t, x) = \widehat{\psi}(t, \widehat{z}(t, x)),\ &\text{is the unique solution of } (4.85), \\ &\text{where } \widehat{z}(t, x) = G(-w(t))[x]. \end{aligned} \tag{4.91}$$

The proof follows the arguments used in Lemma 4.4.4 of section 4.4.3.

Lemma 4.4.13. *Under the hypothesis* (4.81)*, consider the continuous and* $\mathcal{F}_t$*-* $\widehat{z}(t, x) = G(-w(t))[x]$, $t \in [0, T]$, $x \in \mathbb{R}^n$*. Then the following SPDE of is valid*

$$\begin{cases} d_t \widehat{z}(t, x) + \displaystyle\sum_{i=1}^{m} \partial_x \widehat{z}(t, x) g_i(x) \widehat{\circ} dw_i(t) = 0,\ t \in [0, T],\ x \in \mathbb{R}^n \\ \widehat{z}(0, x) = x, \end{cases} \tag{4.92}$$

where the "$\widehat{\circ}$*" is computed by*

$$h_i(t, x) \widehat{\circ} dw_i(t) = h_i(t, x) \cdot dw_i(t) - \frac{1}{2} \partial_x h_i(t, x) g_i(x) dt$$

using Ito "$\cdot$*".*

Proof. The conclusion (4.92) is a direct consequence of applying standard rule of associated with $\sigma = w(t) \in \mathbb{R}^m$ and smooth deterministic mapping $H(\sigma)[x] = G(-\sigma)[x]$. In this respect, using $H(\sigma) \circ G(\sigma)[\lambda] = \lambda \in \mathbb{R}^n$ for any $x = G(\sigma)[\lambda]$, we get

$$\begin{cases} \partial_{\sigma_i} H(\sigma)[x] = -\partial_x\{H(\sigma)[x]\}g_i(x),\ \sigma = (\sigma_1, \dots, \sigma_m) \in \mathbb{R}^m,\ x \in \mathbb{R}^n, \\ \partial^2_{\sigma_i}\{H(\sigma)[x]\} = \partial_{\sigma_i}\{\partial_{\sigma_i}\{H(\sigma)[x]\}\} = \partial_{\sigma_i}\{-\partial_x\{H(\sigma)[x]\}g_i(x)\} \\ \qquad\qquad = \partial_x\{\partial_x\{H(\sigma)[x]\}g_i(x)\}g_i(x),\ \sigma \in \mathbb{R}^m,\ x \in \mathbb{R}^n \end{cases} \tag{4.93}$$

for each $i \in \{1, \dots, m\}$. Recall that the standard rule of lead us to SDE

$$d_t\widehat{z}(t,x) = \sum_{i=1}^m \partial_{\sigma_i}\{H(\sigma)[x]\}_{(\sigma=w(t))} \cdot dw_i(t) + \frac{1}{2}\sum_{i=1}^m \partial^2_{\sigma_i}\{H(\sigma)[x]\}_{(\sigma=w(t))}dt, \tag{4.94}$$

for any $t \in [0,T]$, $x \in \mathbb{R}^n$. Rewritting the right hand side of (4.94) (see (4.93)) we get SPDE of parabolic type given in (4.92). □

Lemma 4.4.14. *Assume the hypotheses* (4.81) *and* (4.82) *are fulfilled and consider* $\{\lambda = \psi(t,x) : t \in [0,T], x \in \mathbb{R}^n\}$ *defined in Lemma 4.4.12. Then* $u(t,x) := h(\psi(t,x))$, $t \in [0,T]$, $x \in \mathbb{R}^n$, $h \in (\mathcal{C}^1_b \cap \mathcal{C}^2)(\mathbb{R}^n)$, *satisfies the following nonlinear SPDE*

$$\begin{cases} d_tu(t,x) + \langle \partial_x u(t,x), \sum_{i=1}^m \varphi_i(\psi(t,x)f_i(x))\rangle dt \\ \qquad\qquad + \sum_{i=1}^m \langle \partial_x u(t,x), g_i(x)\rangle \widehat{\circ} dw_i(t) = 0,\ t \in [0,T] \\ u(0,x) = h(x) \end{cases} \tag{4.95}$$

where the nonstandard "$\widehat{\circ}$" is computed by

$$h_i(t,x)\widehat{\circ}dw_i(t) = h_i(t,x) \cdot dw_i(t) - \frac{1}{2}\partial_x h_i(t,x)g_i(x)dt.$$

The proof uses the same arguments as in Lemma 4.4.6 of section 4.4.3.

Theorem 4.4.15. *Assume that the vector fields* $\{f_1, \dots, f_m\} \subset (\mathcal{C}_b \cap \mathcal{C}^1_b \cap \mathcal{C}^2)(\mathbb{R}^n; \mathbb{R}^n)$, $\{g_1, \dots, g_m\} \subset (\mathcal{C}^1_b \cap \mathcal{C}^2)(\mathbb{R}^n; \mathbb{R}^n)$ *and scalar functions* $\{\varphi_1, \dots, \varphi_m\} \subset (\mathcal{C}^1_b \cap \mathcal{C}^2)(\mathbb{R}^n)$ *fulfil the hypotheses 4.81 and 4.82*

Consider the continuous and $\mathcal{F}_t$- $\{\lambda = \psi(t,x) \in \mathbb{R}^n : t \in [0,T], x \in \mathbb{R}^n\}$ *satisfying integral equations* (4.85) *(see Lemma 4.4.12). Then* $\{u(t,x) := h(\psi(t,x)) : t \in [0,T], x \in \mathbb{R}^n\}$ *fulfils nonlinear SPDE of parabolic type* (4.95) *(see Lemma 4.4.14) for each* $h \in (\mathcal{C}^1_b \cap \mathcal{C}^2)(\mathbb{R}^n)$.

4.4.7 Solution for (P2)

As far as SDE (4.83) is a non-markovian system, the evolution of a functional $S(t,x) := Eh(\widehat{x}_\psi(T;t,x))$, $t \in [0,T]$, $x \in \mathbb{R}^n$, for each $h \in (\mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n)$ will be described using the pathwise representation of the conditioned mean values functional

$$v(t,x) := E\{h(\widehat{x}_\psi(T;t,x)) \mid \psi(t,x)\},\ 0 \leqslant t < T,\ x \in \mathbb{R}^n. \tag{4.96}$$

Here $\widehat{x}_\psi(T;t,x)$ can be expressed using the following integral representation

$$\widehat{x}_\psi(T;t,x) = G(w(T)-w(t)) \circ F[(T-t)\varphi(\psi(t,x))](x),\ 0 \leqslant t < T, \tag{4.97}$$

where $G(\sigma)[z]$ and $F(\sigma)[z]$, $\sigma = (\sigma_1,\ldots,\sigma_m) \in \mathbb{R}^m$, $x \in \mathbb{R}^n$, are defined in (P1) (see (4.84)) for $\varphi := (\varphi_1,\ldots,\varphi_m)$. The right hand side of (4.97) is a continuous mapping of the two independent random variables, $z_1 = w(T) - w(t) \in \mathbb{R}^m$ and $z_2 = \psi(t,x) \in \mathbb{R}^n$ ($\mathcal{F}_t$-measurable) for each $0 \leqslant t < T$, $x \in \mathbb{R}^n$.

Using the parameterized random variable

$$y(t,x;\lambda) = G(w(T)-w(t)) \circ F[(T-t)\varphi(\lambda)](x),\ 0 \leqslant t < T \tag{4.98}$$

we may and do compute the functional $v(t,x)$ in (4.96) by

$$v(t,x) = [Eh(y(t,x;\lambda))](\lambda = \psi(t,x)),\ 0 \leqslant t < T,\ x \in \mathbb{R}^n. \tag{4.99}$$

Here, the functional

$$u(t,x;\lambda) = Eh(y(t,x;\lambda)),\ t \in [0,T],\ x \in \mathbb{R}^n, \tag{4.100}$$

satisfies a backward parabolic equation (Kolmogorov‘s equation) for each $\lambda \in \mathbb{R}^n$ and rewrite (4.99) as follows,

$$v(t,x) = u(t,x;\psi(t,x)),\ 0 \leqslant t < T,\ x \in \mathbb{R}^n. \tag{4.101}$$

In conclusion, the functional $S(t,x) = Eh(\widehat{x}_\psi(T;t,x))$ can be reprezented by

$$S(t,x) = E[E\{h(\widehat{x}_\psi(T;t,x)) \mid \psi(t,x)\}] = Eu(t,x;\psi(t,x)) \tag{4.102}$$

for any $0 \leqslant t < T$, $x \in \mathbb{R}^n$, where $\{u(t,x;\lambda) : t \in [0,T], x \in \mathbb{R}^n\}$ satisfies the corresponding backward parabolic equations with parameter $\lambda \in \mathbb{R}^n$,

$$\begin{cases} \partial_t u(t,x;\lambda) + \langle \partial_x u(t,x;\lambda), f(x,\lambda)\rangle + \dfrac{1}{2}\displaystyle\sum_{i=1}^m \langle \partial_x^2 u(t,x;\lambda) g_i(x), g_i(x)\rangle = 0, \\ u(T,x;\lambda) = h(x),\ f(x,\lambda) = \displaystyle\sum_{i=1}^m \varphi_i(\lambda) f_i(x) + \dfrac{1}{2}\sum_{i=1}^m [\partial_x g_i(x)] g_i(x). \end{cases} \tag{4.103}$$

We conclude these remarks by a theorem.

Theorem 4.4.16. *Assume that the vector fields* $\{f_1, \ldots, f_m\} \subset (\mathcal{C}_b \cap \mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n; \mathbb{R}^n)$, $\{g_1, \ldots, g_m\} \subset (\mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n; \mathbb{R}^n)$, *and scalar functions* $\varphi = (\varphi_1, \ldots, \varphi_m) \subset (\mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n)$ *of SDE* (4.83) *fulfil the hypotheses* (4.81) *and* (4.82). *Then the evolution of the functional*

$$S(t,x) := Eh(\widehat{x}_\psi(T; t, x)),\ t \in [0, T],\ x \in \mathbb{R}^n,\ h \in (\mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n) \tag{4.104}$$

can be described as in (4.102), *where* $\{u(t, x; \lambda) : t \in [0, T], x \in \mathbb{R}^n\}$ *satisfies linear backward parabolic equations* (4.103), *for each* $\lambda \in \mathbb{R}^n$.

Final remark. One may wonder about the meaning of the martingale representation associated with the non-markovian functionals $h(\widehat{x}_\psi(T; t, x))$, $h \in (\mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n)$. In this respect, we may use the parameterized functional $\{u(t, x; \lambda) : t \in [0, T], x \in \mathbb{R}^n\}$ fulfilling backward parabolic equations (4.103). Write

$$h(\widehat{x}_\psi(T; t, x)) = u(T, \widehat{x}_\psi(T; t, x); \widehat{\lambda} = \psi(t, x)) \tag{4.105}$$

and apply the standard rule of stochastic derivation associated with smooth mapping $\{u(s, x; \widehat{\lambda}) : s \in [0, T], x \in \mathbb{R}^n\}$ and stochastic process $\{\widehat{x}_\psi(s; t, x) : s \in [t, T]\}$. We get

$$\begin{aligned} h(\widehat{x}_\psi(T; t, x)) =& u(t, x; \widehat{\lambda}) + \int_t^T (\partial_s + L_{\widehat{\lambda}})(u)(s, \widehat{x}_\psi(s; t, x); \widehat{\lambda}) ds \\ &+ \sum_{i=1}^m \int_t^T \langle \partial_x u(s, \widehat{x}_\psi(s; t, x); \widehat{\lambda}), g_i(x) \rangle dw_i(s), \end{aligned} \tag{4.106}$$

where $L_{\widehat{\lambda}}(u)(s, x; \widehat{\lambda}) := \langle \partial_x u(s, x; \widehat{\lambda}), f(x, \widehat{\lambda}) \rangle + \frac{1}{2} \sum_{i=1}^m \langle \partial_x^2 u(s, x; \widehat{\lambda}) g_i(x), g_i(x) \rangle$ coincides with parabolic operator in PDE (4.103). Using (4.103) for $\widehat{\lambda} = \psi(t, x)$, we obtain the following martingale representation

$$h(\widehat{x}_\psi(T; t, x)) = u(t, x; \psi(t, x)) + \sum_{i=1}^m \int_t^T \langle \partial_x u(s, \widehat{x}_\psi(s; t, x); \widehat{\lambda}), g_i(x) \rangle \cdot dw_i(s), \tag{4.107}$$

which shows that the standard constant in the markovian case is replaced by a $\mathcal{F}_t$-measurable random variable $u(t, x; \psi(t, x))$. In addition, the backward evolution of stochastic functional $\{Q(t, x) := h(\widehat{x}_\psi(T; t, x)) : t \in [0, T], x \in \mathbb{R}^n\}$ given in (4.107) depends essentially on the forward evolution process $\{\psi(t, x)\}$ for each $t \in [0, T]$ and $x \in \mathbb{R}^n$.

Bibliographical Comments

The writing of this part has much in common with the references [1] and [11].

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